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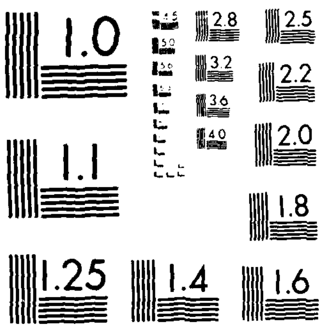

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MULTIPLE SOLUTIONS AND BIFURCATION FOR A CLASS OF NONLINEAR STURM-LIOUVILLE EIGENVALUE PROBLEMS ON AN UNBOUNDED DOMAIN

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ABSTRACT

A class of nonlinear Sturm-Liouville problems is considered. These problems admit zero as a trivial solution and the nonlinear operator linearized about zero has a purely continuous spectrum $[0,\infty)$. Variational techniques and approximation arguments are used to obtain the existence of nontrivial solutions with any prescribed number of nodes and for some nonlinearities it is shown that this solution is unique. Moreover, the lowest point of the continuous spectrum is a bifurcation point; infinitely many continua of solutions, which are distinguished by nodal properties, bifurcate from the line of trivial solutions at this point. Results are also obtained in higher dimensions via investigation of the set of radial solutions of appropriate partial differential equations.

AMS (MOS) Subject Classifications: 34B15, 35B32, 35J20, 35J25

Key Words: bifurcation, nonlinear eigenvalue problem, variational methods, nodal property, radial solutions

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§0. INTRODUCTION

Bifurcation questions for nonlinear elliptic eigenvalue problems on unbounded domains have recently been studied by various authors ([1]-[11],[13],[23],[28],[29],[42]). They have found that the lowest point of the continuous spectrum of the linearized operator is a potential bifurcation point. In these papers, there are mainly two kinds of bifurcation phenomena that have been dealt with: (i) bifurcation of solutions having parameter values in the continuous spectrum ([1]-[8]) and (ii) bifurcation of solutions having parameter values not in the continuous spectrum ([9]-[11]).

In this thesis we further study such problems pertaining to (i). We consider a nonlinear Sturm-Liouville eigenvalue problem for a family of ordinary differential equations and a related class of partial differential equations. In the ordinary differential equation case, we study the boundary value problem

$$-u'' = \lambda r(x)u - F(x,u)u, \quad 0 < x < +\infty \tag{0.1.a}$$

$$u(0)\cos\theta - u'(0)\sin\theta = 0, \quad u \in L^2(0,\infty) \tag{0.1.b}$$

where r and F are nonnegative continuous functions. $F(x,0) = 0$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. The related problem in the partial differential equation case is

$$-\Delta u = \lambda r(x)u - F(x,u)u \quad x \in R^N \tag{0.2.a}$$

$$u \in L^2(R^N) \ . \tag{0.2.b}$$

Küpper [1], [2] first pointed out that a minimal growth condition with respect to x for the nonlinearity is needed to ensure the existence of an L^2-solution of (0.1). In particular, if $F(x,y) = \omega(x)|y|^\sigma$, then there exists a nontrivial L^2-solution if and only if $\int_0^\infty \omega^{-2/\sigma}dx < +\infty$. The point $\lambda = 0$ is the infimum of the continuous spectrum. Küpper proved that for any $\lambda > 0$ there exists a positive solution and that these solutions form a continuum bifurcating from $(\lambda,u) = (0,0)$. By a continuum of solutions, we mean a set of pairs $(\lambda,u) \in \mathbb{R} \times E$, satisfying (0.1), which is connected with respect to a reasonable topology in a function space E associated with (0.1).

Applying arguments due to Ljusternik and Schnirelman, Bongers, Heinz and Küpper [3] considered Dirichlet boundary value problems for both ordinary differential equation and partial differential equation. They proved in particular for problems like (0.1) and (0.2) that for every $r > 0$, there exists a sequence $(\lambda_k^{(r)}, u_k^{(r)})_{k\geq 1}$ of solutions such that $\|u_k^{(r)}\|_{L^2} = r$ and that $\lim_{r\to\infty} \lambda_k^{(r)} = \infty$, while $\lim_{r\to 0^+} \lambda_k^{(r)} = 0$. In the ODE case, Heinz [6] further related the Ljusternik-Schnirelman critical levels associated with (0.1) to nodal properties of solutions. This work [3] shows the problem has a sequence of solution "branches" emanating from $(\lambda,u) = (0,0)$ and a natural open question is whether these "branches" are connected.

Jones and Küpper [4] studied a more restricted problem

$$-u'' = \lambda u - \omega(x)|u|^\sigma u, \qquad 0 < x < +\infty \tag{0.3.a}$$

$$u(0) = 0, \qquad u \in L^2[0,\infty) \,. \tag{0.3.b}$$

They used various assumptions on ω near infinity, a good model case being $\omega(x) = p(x)e^{\alpha x}$ with $\alpha > 0$ and $p(x) > 0$ a polynomial. For each $\lambda > 0$, they employed phase portrait techniques to construct a sequence $(u_{k,\lambda})_{k\geq 1}$ of solutions such that $u_{k,\lambda}$ has exactly $k - 1$ distinct interior zeroes. Moreover, $u_{k,\lambda} \to 0$ as $\lambda \to 0$.

More recently, Heinz [7] treated (0.3) imposing the following hypotheses:

1° $\log \omega$ is convex

2° ω is monotonically nondecreasing

3° ω' attains positive values

and proved for fixed λ that L^2-solutions with a fixed number of zeroes cannot be the limit point of L^2-solutions which possess the same nodal property. Thus, among these L^2-solutions having the same nodal property, there is one, having a minimal (in magnitude) derivative at $x = 0$, which he called the "preferred solution". He showed the "preferred solutions" form connected sets and all of these continua emanate from the point $(0,0) \in \mathbf{R} \times L^2$ in some suitable norm. Then, in [8], he used arguments of Ljusternik-Schnirelman type to get further existence results for solutions of (0.1).

In this thesis, we will investigate various questions for problem (0.1):

(a) Existence and uniqueness of positive and negative solutions.

(b) Existence and uniqueness of solutions with a prescribed number of nodes.

(c) Bifurcation of connected sets of solutions which possess nodal properties.

In §1, we will mainly aim at question (a) for (0.1). Under the following assumptions:

(r.1) $r \in C([0,\infty),(0,\infty))$ $0 < r_1 < r(x) < r_2 < +\infty$ for $x \in [0,\infty)$.

(F.1) $F : [0,\infty) \times R \to [0,\infty)$ is continuous.

(F.2) There exist positive numbers σ_i and continuous functions $\omega_i : [0,\infty) \to (0,\infty)$ which satisfying $\int_0^\infty \omega_i^{-2/\sigma_i} dx < +\infty$, $i = 1,2$ such that $F(x,y) > \omega_1(x)|y|^{\sigma_1}$ for $x \in [0,\infty)$, $y > 0$ and $F(x,y) > \omega_2(x)|y|^{\sigma_2}$ for $x \in [0,\infty)$, $y < 0$.

(F.3) $\lim_{|y|\to 0} F(x,y) = 0$ uniformly on compact subsets of $[0,\infty)$.

(F.4) For fixed $x \in [0,\infty)$, $F(x,y)$ is an increasing function of y if $y > 0$ and a decreasing function of y if $y < 0$.

We will prove

Theorem 0.4

Given $\lambda > 0$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, there exists a unique positive (resp. negative) solution u which satisfies (0.1). Moreover, $u \in C^2[0,\infty) \cap H^1[0,\infty)$ and $u(x) \to 0$, $u'(x) \to 0$ as $x \to \infty$.

To obtain the existence, an approximation approach will be used; taking as approximate solutions those for the bounded interval case. Compared to variational methods, our argument has the advantage that

(F.5) $F(x,-y) = F(x,y)$ for $x \in [0,\infty)$, $y \in R$

need not be assumed. In the variational argument used in [3], [6] and [8] the growth condition $\int_0^\infty \omega^{-2/\sigma} dx < +\infty$ is used to give a compact imbedding property. Here, we use it to provide a prior estimates which allow us to pass to the limit from approximate solutions.

Uniqueness will be proved with the aid of several useful "monotonicity" properties for positive and negative solutions derived from the monotonicity assumption (F.4). Also, in the proof of the existence result, "monotonicity" properties will be used to prevent the limit of approximate solutions from degenerating to the trivial solution.

In §2, a method that pieces together alternately positive and negative solutions on adjacent intervals will be used to obtain solutions with a prescribed number of nodes. This idea originated with Nehari [15] for a bounded interval. The same kind of technique was generalized to an unbounded domain by Ryder [16]. Hempel [17] also used such an approach on a rather different class of equations for a bounded domain. Our argument is closely related to Hempel's. However, we treat the problem for the unbounded domain case. Our main result in §2 is:

Theorem 0.5

Assume (r.1), (F.1)-(F.4) and (F.5) are satisfied. Given $\lambda > 0$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. Then for every $n > 1$, there exists a solution which satisfies (0.1), having exactly $n - 1$ zeroes in $(0,\infty)$, and being positive (resp. negative) in a deleted neighborhood of $x = 0$.

In §3, with the help of the "monotonicity" properties of §1, we use the solutions obtained in §2 as starting points and construct an iteration scheme to get the result of Theorem 0.5 under the weaker symmetricity assumption:

(F.5)' There are positive numbers δ and X such that

$$F(x,-y) = F(x,y) \quad \text{for} \quad x \in [X,\infty) \quad \text{and} \quad |y| < \delta.$$

In §4, we consider the problem with a special form for the nonlinearity

$$-u'' = \lambda u - \psi(w(x)|u|^{\sigma})u, \qquad 0 < x < +\infty \tag{0.6.a}$$

$$u(0)\cos\theta - u'(0)\sin\theta = 0, \qquad u \in L^2[0,\infty) \;. \tag{0.6.b}$$

Assuming $\psi(t) > pt^q$, $\psi' > 0$, $w'(0) > 0$ and

(w.1) $\dfrac{w'}{w}$ is nondecreasing on $[0,\infty)$.

We prove the uniqueness of solutions having a prescribed number of nodes. Since the problem Heinz treated in [8] is a special case of (0.6), we solve a question which was left open in [8]. Having this uniqueness result we then show that there are infinitely many continuous curves of solutions for (0.6) which are characterized by their nodal properties. Each curve can be parametrized by the corresponding eigenvalue parameter λ and all these curves bifurcate from (0.0).

In §5, we give a bifurcation result which is applicable to more general nonlinearities. However, the result is weaker than that in §4 in that only connected sets rather than curves of solutions will be obtained.

In §6, we study radial solutions of (0.2). Let ρ denote the radial variable. The growth condition $\int_0^\infty \omega^{-2/\sigma}dx < \infty$ in the one-dimensional case will be replaced by a parallel one, $\int_0^\infty \rho^{N-1}\omega^{-2/\sigma}d\rho < +\infty$ here. If $\hat{\omega}(x) = \omega(\rho)$ for $|x| = \rho$, $x \in \mathbb{R}^N$, this is equivalent to $\int_{\mathbb{R}^N} \hat{\omega}^{-2/\sigma}dx < +\infty$ which has been used in [3] and [8] although they considered general solutions rather than radial ones. However, looking for radial solutions allows us to pursue solutions with nodal properties. The main new difficulty here is the occurrence of a singularity in the equation at the *origin*. We overcome this difficulty by making an additional approximation. Using a transformation of variables and results established in previous sections, we obtain analogous results to those in the one-dimensional case.

§1. EXISTENCE AND UNIQUENESS OF POSITIVE AND NEGATIVE SOLUTIONS

In this section, we will obtain existence and uniqueness results for positive and negative solutions for the following problems

$$(\mathrm{I})_a \quad \begin{cases} -u'' = \lambda r(x)u - F(x,u)u, \quad a < x < +\infty & (1.1.a) \\ u(a)\cos\theta - u'(a)\sin\theta = 0, \quad u \in L^2[a,\infty) & (1.1.b) \end{cases}$$

where $a > 0$ and $0 \le \theta \le \frac{\pi}{2}$. Throughout §1-§5, prime will always denote differentiation with respect to the space variable. The functions r and F are assumed to satisfy

(r.1) $r \in C([0,\infty),(0,\infty))$, $0 < r_1 \le r(x) \le r_2 < +\infty$ for $x \in [0,\infty)$.

(F.1) $F : [0,\infty) \times R \to [0,\infty)$ is continuous.

(F.2) There exist positive numbers σ_i and continuous functions $\omega_i : [0,\infty) \to (0,\infty)$ which satisfy $\int_0^\infty \omega_i^{-2/\sigma_i} dx < +\infty$, $i = 1,2$, such that $F(x,y) \ge \omega_1(x)|y|^{\sigma_1}$ for $x \in [0,\infty)$, $y > 0$ and $F(x,y) \ge \omega_2(x)|y|^{\sigma_2}$ for $x \in [0,\infty)$, $y < 0$.

(F.3) $\lim_{|y|\to 0} F(x,y) = 0$ uniformly on compact subsets of $[0,\infty)$.

(F.4) For fixed $x \in [0,\infty)$, $F(x,y)$ is an increasing function of y if $y > 0$ and a decreasing function of y if $y < 0$.

By a solution of $(\mathrm{I})_a$ we mean $u \in C^2[a,\infty) \cap H^1[a,\infty)$ which satisfies (1.1).

Now, we state the main existence result for positive and negative solutions:

Theorem 1.2

Suppose (r.1), (F.1)-(F.4) are satisfied. Given $\lambda > 0$, $a > 0$ and $0 \le \theta \le \frac{\pi}{2}$ there exists a positive (resp. negative) solution u

which satisfies $(I)_a$ and

$$\lim_{x\to\infty} u(x) = 0, \qquad \lim_{x\to\infty} u'(x) = 0 .$$

Remark 1.3

The existence result of positive and negative solutions has been obtained via different methods ([1]-[8]). However, they only treated the boundary condition $\theta = 0$ in (1.1.b). In [3], [6] and [8] the authors imposed the minimal growth condition on the function $g(x,y) = \frac{\partial f(x,y)}{\partial y}$ where $f(x,y) = F(x,y)y$. It is easy to see if $F(x,y) > \omega(x)|y|^\sigma$ and $\frac{\partial f}{\partial y}$ exists then, by (F.4), $g(x,y) = F(x,y) + y \cdot \frac{\partial F(x,y)}{\partial y} > F(x,y) > \omega(x)|y|^\sigma$. Also, they assumed an upper bound for $g(x,y)$ which we do not need.

Our strategy is to approximate solutions of $(I)_a$ by those of

$$(I)_{a,b} \quad \begin{cases} -u'' = \lambda r(x)u - F(x,u)u & (1.4.a) \\ u(a)\cos\theta - u'(a)\sin\theta = 0, \quad u(b) = 0 . & (1.4.b) \end{cases}$$

Existence results for (1.4) have already been established in the literature (e.g. [24]). A partial uniqueness result is known for (1.1). The more general form we require is:

Theorem 1.5

Suppose (r.1), (F.1) and (F.4) are satisfied. Let $\lambda > 0$, $a > 0$ and $0 < \theta < \frac{\pi}{2}$ be fixed. If u_1, u_2 are two solutions of $(I)_{a,b}$ (resp. $(I)_a$) such that $u_1, u_2 > 0$ or $u_1, u_2 < 0$ on (a,b) (resp. (a,∞)), then

$$u_1 \equiv u_2 \quad \text{in} \quad [a,b] \quad (\text{resp. } [a,\infty)) .$$

Since the proof of Theorem 1.5 will immediately follow from a "monotonicity" lemma, we postpone it till then.

Remark 1.6

(a) For a bounded or unbounded domain with $\theta = 0$ this result has been obtained by several authors (e.g. [1],[3],[4],[6], [7],[17],[22]).

(b) We will let $V_{\pm}(\lambda,a,b,\theta,\cdot)$ (resp. $V_{\pm}(\lambda,a,\infty,\theta,\cdot)$) represent the unique positive and negative solution for $(I)_{a,b}$ (resp. $(I)_a$) respectively. When some of parameters λ,a,b,θ are known or considered fixed they will be suppressed in the above notation.

(c) If the function F in (1.1.a) is further assumed to satisfy $F(x,-y) = F(x,y)$ for $x \in [0,\infty)$, $y \in R$, then it is clear that $V_- = -V_+$.

To prove Theorems 1.2 and 1.5, we need some preliminaries which including technical results, "monotonicity" lemmas and estimates of solutions for $(I)_a$ and $(I)_{a,b}$. We first state a result of Wintner and Hartman [21].

Lemma 1.7

Let φ_1 and φ_2 be continuous functions on $[a,\infty)$ such that φ_1 is bounded from above and $\varphi_2 \in L^2[a,\infty)$. If u is a solution of the differential equation $u''(x) + \varphi_1(x)u(x) = \varphi_2(x)$ and $u \in L^2[a,\infty)$ then $u \in H^1[a,\infty)$ and $u(x) \to 0$, $u'(x) \to 0$ as $x \to \infty$.

Next, we prove two technical lemmas.

Lemma 1.8

Suppose $\varphi_1,\varphi_2 \in C[\alpha,\beta]$ (resp. $[\alpha,\infty)$). Then there are no functions $u,v \in C^2[\alpha,\beta]$ (resp. $[\alpha,\infty)$) satisfying

$$u(\alpha)v'(\alpha) - u'(\alpha)v(\alpha) < 0 \tag{1.9}$$

$$u(\beta)v'(\beta)-u'(\beta)v(\beta) \geq 0 \text{ (resp. } \lim_{\beta\to\infty} u(\beta)v'(\beta)-u'(\beta)v(\beta) \geq 0) \tag{1.10}$$

and for $x \in (\alpha,\beta)$ (resp. (α,∞))

$$u \cdot v > 0$$

$$-u'' \leq \varphi_1(x)u \tag{1.11}$$

$$-v'' \geq \varphi_2(x)v \tag{1.12}$$

$$\varphi_2 \geq \varphi_1 \,. \tag{1.13}$$

Moreover, at least one of inequalities (1.9), (1.10) is strict or at least one of (1.11)-(1.13) has strict inequality on a subinterval of (α,β) (resp. (α,∞)).

Proof

If $u,v > 0$ multiplying (1.11) by $-v$ and (1.12) by u and adding together, we obtain

$$u''v - v''u \geq (\varphi_2(x) - \varphi_1(x))uv \,. \tag{1.14}$$

Since $u''v - v''u = (u'v - v'u)'$ by integrating (1.14) on $[\alpha,\beta]$, we have

$$u'(\beta)v(\beta) - v'(\beta)u(\beta) - u'(\alpha)v(\alpha) + v'(\alpha)u(\alpha)$$

$$\geq \int_\alpha^\beta (\varphi_2(x)-\varphi_1(x))uvdx \,. \tag{1.15}$$

From (1.9) and (1.10) we know the left-hand side of (1.15) is nonpositive and is negative if at least one of inequalities (1.9), (1.10) is strict. On the other hand, the right-hand side of (1.15) is nonnegative and is positive if $\varphi_2(x) > \varphi_1(x)$ on a subinterval of $[\alpha,\beta]$. Finally, note that the inequality of (1.15) is strict if (1.11) or (1.12) is strict on a subinterval of $[\alpha,\beta]$. Therefore, if one of the above cases occurs, we have a contradiction to (1.15).

In the case of $[\alpha,\infty)$, the proof is the same except for letting $\beta \to \infty$ in (1.15). The proof for the case $u,v < 0$ is similar. We omit it.

Corollary 1.16

Suppose $\varphi_1,\varphi_2 \in C[\alpha,\beta]$ (resp. $[\alpha,\infty)$) and let $u,v \in C^2[\alpha,\beta]$ (resp. $[\alpha,\infty)$) such that $u(\alpha) = v(\alpha)$ and $u(\beta) = v(\beta)$ (resp. $\lim_{\beta\to\infty} u(\beta)v'(\beta) - u'(\beta)v(\beta) = 0$). If, for $x \in (\alpha,\beta)$ (resp. (α,∞)), we have (1.11)-(1.13) and one of them has strict inequality on a subinterval of (α,β) (resp. (α,∞)). Then neither

(i) $u(x) > v(x) > 0$ for $x \in [\alpha,\beta]$ (resp. $[\alpha,\infty)$)

nor

(ii) $v(x) < u(x) < 0$ for $x \in [\alpha,\beta)$ (resp. $[\alpha,\infty)$)

can occur.

Proof

Suppose (i) occurs, then $u'(\alpha) > v'(\alpha)$ and $u'(\beta) < v'(\beta)$ in the case of $[\alpha,\beta]$, hence (1.9) and (1.10) are satisfied and a contradiction immediately follows from Lemma 1.8. An analogous argument takes care of (ii). We omit it.

Having these lemmas as consequences, we are going to establish several "monotonicity" properties of positive and negative solutions for $(I)_{a,b}$ and $(I)_a$. In the remainder of this section, when the proof for the positive solutions is the same as that for the negative solutions we will only carry out the former.

Corollary 1.17

Let u and v be positive (resp. negative) solutions of (1.18) and (1.19) respectively:

$$-u'' = \lambda r(x)u - F(x,u)u \tag{1.18}$$

$$-v'' = \mu s(x)v - H(x,v)v . \tag{1.19}$$

Suppose the functions r,s and F,H satisfy, respectively, (r.1) and (F.1), (F.4). If

$$\lambda r \leq \mu s \tag{1.20}$$

and

$$F \geq H \tag{1.21}$$

then

(i) If u and v satisfy (1.4.b), we have

$$|u(x)| \leq |v(x)| \quad \text{for} \quad x \in (a,b) \quad \text{if} \quad \theta = 0 \quad \text{and for}$$
$$x \in [a,b) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2} , \tag{1.22}$$

$$|u'(b)| \leq |v'(b)| \quad \text{and} \quad |u'(a)| \leq |v'(a)| . \tag{1.23}$$

(ii) If u and v satisfy (1.1.b), we have

$$|u(x)| \leq |v(x)| \quad \text{for} \quad x \in (a,\infty) \quad \text{if} \quad \theta = 0 \quad \text{and for}$$
$$x \in [a,\infty) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2} , \tag{1.24}$$

and $$|u'(a)| \leq |v'(a)| . \tag{1.25}$$

Moreover, if the inequality (1.20) is strict or $F(x,y) > H(x,y)$ for $y \neq 0$ then inequalities (1.22) and (1.24) are strict.

Remark 1.26

This generalize results of Küpper [1] and Heinz [6] where they treated the case $\theta = 0$ and obtained the "monotonicity" of solutions with respect to the eigenvalue parameter λ.

<u>Proof</u>

(i) Suppose $u(t) > v(t) > 0$ for some $t \in (a,b)$. By the continuity of u and v and the boundary condition (1.4.b) we know there is a subinterval (α,β) of (a,b) such that either

1° $u(x) > v(x) > 0$ for $x \in (\alpha,\beta)$ and

$u(\alpha) = v(\alpha)$, $u(\beta) = v(\beta)$

or

2° $\alpha = a$, $u(x) > v(x) > 0$ for $x \in [a,\beta)$, $u(\beta) = v(\beta)$.

Suppose first that 1° prevails. Let $\varphi_1(x) = \lambda r(x) - F(x,u(x))$, $\varphi_2(x) = \mu s(x) - H(x,v(x))$ for $x \in [\alpha,\beta]$. Then it follows from (F.4) that $F(x,u(x)) > F(x,v(x))$ for $x \in [\alpha,\beta]$. Thus, together with that $\lambda r < \mu s$ and $F > H$, we have

$$\varphi_2(x) > \varphi_1(x) \quad \text{for} \quad x \in [\alpha,\beta] \; . \tag{1.27}$$

Applying Corollary 1.16, we get a contradiction. Therefore 1° is not possible.

Next suppose that 2° holds. Arguing like the beginning of the proof of Lemma 1.8, with the same φ_1, φ_2 as defined above, we obtain

$$u'(\beta)v(\beta) - v'(\beta)u(\beta) - u'(a)v(a) + v'(a)u(a)$$

$$= \int_a^\beta (\varphi_2(x) - \varphi_1(x))uv \; . \tag{1.28}$$

From (1.4.b), we have

$$u'(a)v(a) - v'(a)u(a) = 0 \tag{1.29}$$

and it is clear that

$$u(\beta)v'(\beta) - u'(\beta)v(\beta) > 0 \; .$$

Applying Lemma 1.8, we conclude that 2° is also impossible.

Therefore (1.22) must be valid, and (1.23) immediately follows from (1.22). Moreover, if $u(t) = v(t)$ for $t \in (a,b)$ in the case $\theta = 0$ and for $t \in [a,b)$ in the case $0 < \theta < \frac{\pi}{2}$, then letting $z = u - v$, we have $z(t) = 0$ and $z(x) < 0$ for $x \in [a,b]$. Thus z has a maximum at t. However, from (1.18) and (1.19)

$$z''(t) = [F(t,u(t)) - H(t,u(t)) + \mu s(t) - \lambda r(t)]u(t)$$

which is positive if the inequality (1.20) is strict or $F(x,y) > H(x,y)$ for $y \neq 0$. This contradiction indicates the inequality (1.22) in this situation must be strict.

(ii) Suppose again that $u(t) > v(t) > 0$ for some $t \in [a,\infty)$, by the boundary conditions $u,v \in L^2[a,\infty)$ and Lemma 1.7 we know that $u(x) \to 0$ and $v(x) \to 0$ as $x \to +\infty$. Hence by the continuity and boundary conditions (1.1.b), there is a subinterval (α,β) of (a,∞) such that either 1° or 2° as in (i) occurs except that now β could be $+\infty$.

In case β is finite, the proof is contained in (i). Thus we consider the situation that $\beta = +\infty$. With the same φ_1,φ_2 as above and by the same reasoning as shown in (1.27), we have

$$\varphi_2 > \varphi_1 \quad \text{for} \quad x \in [a,\infty) . \tag{1.30}$$

From Lemma 1.7, it is easy to see that

$$\lim_{x\to\infty} u(x)v'(x) - u'(x)v(x) = 0 . \tag{1.31}$$

Hence it is clear that 1° is contrary to Corollary 1.16. Therefore, we consider 2°. From (1.1.b), we get (1.29). This together with (1.30) and (1.31) contrary to Lemma 1.8.

Therefore (1.24) must hold and the last assertion implies (1.25). Finally, the same argument as in (i) shows the inequality (1.24) is strict provided that the inequality (1.20) is strict or $F(x,y) > H(x,y)$ for $y \neq 0$.

<u>Proof of Theorem 1.5</u>

Let u and v be positive solutions for $(I)_{a,b}$ (resp. $(I)_a$), then it follows from (1.22) (resp. (1.24)) that $u > v$ and $v > u$. Therefore, $u \equiv v$.

<u>Corollary 1.32</u>

Let u and v be positive (resp. negative) solutions of equation (1.1) and satisfy, respectively, the boundary conditions either

(i) $u(a)\cos\theta_1 - u'(a)\sin\theta_1 = 0, \quad u(b) = 0$ (1.33)

$v(a)\cos\theta_2 - v'(a)\sin\theta_2 = 0, \quad v(b) = 0$ (1.34)

or

(ii) $u(a)\cos\theta_1 - u'(a)\sin\theta_1 = 0, \quad u \in L^2[a,\infty)$ (1.35)

$v(a)\cos\theta_2 - v'(a)\sin\theta_2 = 0, \quad v \in L^2[a,\infty)$. (1.36)

Assume (r.1), (F.1) and (F.4) are satisfied. If $0 < \theta_1 < \theta_2 < \pi$ then

$$|u(x)| < |v(x)| \quad \text{for} \quad x \in [a,b) \quad \text{in case (i)} \tag{1.37}$$

or

$$|u(x)| < |v(x)| \quad \text{for} \quad x \in [a,\infty) \quad \text{in case (ii)} . \tag{1.38}$$

Moreover, if $\theta_2 > \theta_1$ and we further assume

(f.1) $f(x,y)$ is locally Lipschitz continuous in y

then inequalities (1.37) and (1.38) are strict.

Proof

(i) Suppose u,v are positive solutions and $u(t) > v(t) > 0$ for some $t \in [a,b)$. Arguing as in the beginning of the proof of Corollary 1.17 (i), we proceed to the situation 1° or 2° as in there. Let $\varphi_1(x) = \lambda r(x) - F(x,u(x))$, $\varphi_2(x) = \lambda r(x) - F(x,v(x))$, it follows from the assumption (F.4) that $\varphi_2 > \varphi_1$ and hence 1° violates Corollary 1.16.

To consider 2°, suppose first $0 = \theta_1 < \theta_2 < \pi$. From $\theta_1 = 0$ and (1.33), we know $u(a) = 0$ and $u'(a) > 0$. Since $v(a) > 0$,

$$u(a)v'(a) = 0 < v(a)u'(a) .$$

If $0 < \theta_1 < \theta_2 < \pi$, (1.33) and (1.34) are equivalent to

$$\frac{u'(a)}{u(a)} = \cot\theta_1 ,$$

and

$$\frac{v'(a)}{v(a)} = \cot\theta_2 .$$

Since $\cot\theta_1 > \cot\theta_2$, we get

$$u(a)v'(a) < v(a)u'(a) . \tag{1.39}$$

Thus, we have (1.39) in either case. Also, since $u(x) > v(x) > 0$, for $x \in [a,\beta)$ and $u(\beta) = v(\beta)$, $v'(\beta) > u'(\beta)$. Hence

$$u(\beta)v'(\beta) - u'(\beta)v(\beta) > 0 .$$

But then Lemma 1.8 shows this is not possible and hence $u < v$.

To prove the final assertion, suppose $u(s) = v(s)$ for some $s \in (a,b)$. Then (1.37) implies $u'(s) = v'(s)$. If (f.1) is satisfied, the basic existence-uniqueness theorem for the initial value problem tells us $u \equiv v$ which is obviously absurd due to their different initial conditions at a. Thus $v(x) > u(x)$ for $x \in (a,b)$.

Next, if $u(a) = v(a) > 0$, then (1.33), (1.34) together with $0 < \theta_1 < \theta_2 < \pi$ imply $u'(a) > v'(a)$, however (1.37) tells us that $u'(a) < v'(a)$ which leads a contradiction. Finally, if $u(a) = v(a) = 0$ it follows from $\theta_2 > 0$ that $v'(a) = 0$. Then $v \equiv 0$ via (f.1), contrary to hypothesis.

(ii) The proof is the same as in (i) except for handling the case $\beta = +\infty$. This is easily carried out, using Lemma 1.7, as in the proof of Corollary 1.17 (ii).

Before continuing giving more "monotonicity" properties of solutions, we quote a known existence result for (1.4). Let $\mu_n = \mu_n(a,b,\theta)$, $(n = 1,2,3,\ldots)$ be the n-th eigenvalue of

$$-v'' = \lambda r(x)v, \qquad a < x < b \tag{1.40.a}$$

$$v(a)\cos\theta - v'(a)\sin\theta = 0, \qquad v(b) = 0\ , \tag{1.40.b}$$

the linearized equation of $(I)_{a,b}$ linearized about the trivial solution $u \equiv 0$. It is well-known (see [27]) that

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots\ , \tag{1.41.a}$$

$$\lim_{n\to\infty} \mu_n(a,b,\theta) = +\infty\ , \tag{1.41.b}$$

the functions μ_n are continuous in a, b and θ, and for fixed a and θ, the μ_n are decreasing functions of b such that

$$\lim_{b\to a^+} \mu_n(a,b,\theta) = +\infty\ , \tag{1.41.c}$$

and

$$\lim_{b \to \infty} \mu_n(a,b,\theta) = 0 \ . \qquad (1.41.d)$$

Moreover, if a and b are fixed, μ_n are decreasing functions of θ for $0 < \theta < \frac{\pi}{2}$.

Remark 1.42

We use the notation $\mu_n(a,b)$ when θ is known or considered fixed and simply write μ_n if a, b and θ all are fixed.

We denote by $S^+_{a,b,n}(\lambda,\theta)$ (resp. $S^-_{a,b,n}(\lambda,\theta)$) the sets of $u \in C'[a,b]$ such that u satisfies (1.4), $u > 0$ (resp. < 0) in a deleted neighborhood of $x = a$, u has exactly $n - 1$ simple zeroes in (a,b) where $\lambda > 0$, $0 < \theta < \frac{\pi}{2}$ and $n > 1$ is an integer.

Proposition 1.43

Suppose (r.1), (F.1)-(F.3) are satisfied. Let $0 < \theta < \frac{\pi}{2}$ be fixed

(i) If $\lambda < \mu_1$ and u is a solution of $(I)_{a,b}$ then $u \equiv 0$.

(ii) If $\lambda < \mu_n$ and u is a solution of $(I)_{a,b}$ then $u \notin S^{\pm}_{a,b,n}(\lambda,\theta)$.

(iii) For any $\lambda > \mu_n$, $S^+_{a,b,n}(\lambda,\theta) \neq \emptyset$ and $S^-_{a,b,n}(\lambda,\theta) \neq \emptyset$.

Remark 1.44

Proposition 1.43 is actually a special case of a more general result in [25] and (F.2) can be replaced by any assumption which insures that $F(x,y) \to \infty$ as $|y| \to +\infty$.

Corollary 1.45

Suppose (r.1), (F.1)-(F.4) are satisfied

(i) Let $0 < \theta < \frac{\pi}{2}$ be fixed in (1.1.b) and (1.4.b). If $\lambda > \mu_1(a,b,\theta)$ and $b < b_1 < +\infty$ then for $x \in (a,b)$

$$|V_{\pm}(\lambda,a,b,x)| \le |V_{\pm}(\lambda,a,b_1,x)| \ . \tag{1.46}$$

If $0 < \theta < \frac{\pi}{2}$, (1.46) holds for $x \in [a,b)$. If $0 < \theta < \frac{\pi}{2}$

$$|V'_{\pm}(\lambda,a,b,a)| \le |V'_{\pm}(\lambda,a,b_1,a)| \ . \tag{1.47}$$

(ii) Let $\theta = 0$ in (1.1.b) and (1.4.b). If $\lambda > \mu_1(a,b,0)$ and $a_1 < a$ then for $x \in [a,b)$

$$|V_{\pm}(\lambda,a,b,x)| \le |V_{\pm}(\lambda,a_1,b,x)| \tag{1.48}$$

and

$$|V'_{\pm}(\lambda,a,b,b)| \le |V'_{\pm}(\lambda,a_1,b,b)| \ . \tag{1.49}$$

If $\lambda > 0$ and $a_1 < a$ then for $x \in [a,\infty)$

$$|V_{\pm}(\lambda,a,\infty,x)| \le |V_{\pm}(\lambda,a_1,\infty,x)| \ . \tag{1.50}$$

(iii) Let $0 < \theta < \frac{\pi}{2}$ be fixed in (1.1.b) and (1.4.b) and $\lambda > \mu_1(a,b,\theta)$. Let $a_1 < a$ and $F_1(x,y) = F(x,y)/r(x)$. If for all fixed $y \neq 0$, $F_1(x,y)$ is nondecreasing in x, then (1.48)-(1.50) hold.

(iv) Let $\theta = 0$ in (1.1.b), (1.4.b). If $\lambda > \mu_1(a,b,0)$ and $a_1 < a < b < b_1 \le +\infty$ then for $x \in [a,b]$

$$|V_{\pm}(\lambda,a,b,x)| \le |V_{\pm}(\lambda,a_1,b_1,x)| \ . \tag{1.51}$$

(v) Let $0 < \theta < \frac{\pi}{2}$ be fixed in (1.1.b), (1.4.b). Suppose for fixed $y \neq 0$, $F_1(x,y)$ is nondecreasing in x. If $\lambda > \mu_1(a,b,\theta)$ and $a_1 < a < b < b_1 \le +\infty$, then for $x \in [a,b]$ (1.51) holds.

Moreover, if (f.1) is further assumed, all inequalities (1.46)-(1.51) except for the inequality (1.50) of (iii) in the case $\theta = \frac{\pi}{2}$ are strict. The exceptional one is also strict provided that in addition to assuming (f.1), $\frac{\partial F_1}{\partial x} > 0$ and $\frac{\partial F_1}{\partial y} > 0$ for $y \neq 0$ are also satisfied.

Remark 1.52

In this corollary, we suppress the dependence on θ from our notation $V_{\pm}$.

Proof

(i) Put $u = V_+(\lambda,a,b,\cdot)$ and $v = V_+(\lambda,a,b_1,\cdot)$. Suppose $u(t) > v(t)$ for some $t \in (a,b)$. By the continuity of u,v the positivity of u,v and the boundary conditions $u(b) = 0$ we proceed to the situation 1° and 2° as in Corollary 1.17 (i). With $\varphi_1(x) = \lambda r(x) - F(x,u(x))$ and $\varphi_2(x) = \lambda r(x) - F(x,v(x))$ the same proof as in there shows both 1° and 2° are impossible. Thus (1.46) holds and hence (1.47) follows.

(ii) Set $u = V_+(\lambda,a,b,\cdot)$ and $v = V_+(\lambda,a_1,b,\cdot)$. Suppose $u(t) > v(t)$ for some $t \in (a,b)$. Then the boundary conditions $u(a) = u(b) = 0$ together with the positivity and continuity of u,v imply the situation 1° as above must occur which is contrary to Corollary 1.16. Thus (1.48) holds and (1.49) consequently follows. Since the proof of (1.50) is the same as above, except for treating boundary conditions at infinity, which has been done before, we skip it.

(iii) Let $u = V_+(\lambda,a,b,\cdot)$, $v = V_+(\lambda,a_1,b,\cdot)$. It is known (see e.g. [24], Chap. 4) that v can not have a double zero, that is, we have $v^2(x) + v'^2(x) \neq 0$. Hence the Prüfer substitution ([40], Chap. 10) can be made as follows. Define $\rho(x) = v^2(x) + v'^2(x)$, $\tau(x) = \arctan\left(\frac{v(x)}{v'(x)}\right)$, then $v(x) = \rho(x)\sin\tau(x)$,

$v'(x) = \rho(x)\cos\tau(x)$, $\tau(a_1) = \theta$ and

$$\frac{d\tau}{dx} = [\lambda r(x) - F(x,v(x))]\sin^2\tau(x) + \cos^2\tau(x) \ .$$

It is easy to see that

$$\text{If} \quad \tau(t) \in \left[0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right] \quad \text{and} \quad F_1(t,v(t)) < \lambda,$$

$$\text{or} \quad \tau(t) = \frac{\pi}{2} \quad \text{and} \quad F_1(t,v(t)) < \lambda, \quad \text{then} \quad \frac{d\tau(t)}{dx} > 0 \ . \qquad (1.53.a)$$

Also, since $v(x)$ is a positive solution, $\tau(t) \in (0,\pi)$ for $t \in (a_1,b)$. We claim

$$\text{If} \quad F_1(t,v(t)) > \lambda \quad \text{then} \quad \tau(t) \notin \left[0, \tfrac{\pi}{2}\right] \ . \qquad (1.53.b)$$

$$\text{If} \quad F_1(t,v(t)) \geq \lambda \quad \text{then} \quad \tau(t) \notin \left[0, \tfrac{\pi}{2}\right) \ . \qquad (1.53.c)$$

Indeed, by (1.1.a) $F_1(t,v(t)) > \lambda$ implies $v''(t) > 0$. Suppose $\tau(t) \in \left[0, \frac{\pi}{2}\right]$ then $v'(t) \geq 0$. Hence $v'(x) > 0$ for $x \in (t,t + \varepsilon)$ with some $\varepsilon > 0$. Since $v(t) \geq 0$, $v(x) > v(t) \geq 0$ for $x \in (t,t + \varepsilon]$. Suppose there is an $\varepsilon_0 > 0$ such that

$$\varepsilon_0 = \sup\{\varepsilon \,|\, v'(x) > 0 \quad \text{for} \quad x \in (t,t + \varepsilon)\} \ .$$

Then, since $F_1(x,y)$ is nondecreasing in x and, by (F.4), is increasing in y, $F_1(t + \varepsilon_0, v(t + \varepsilon_0)) > F(t,v(t)) > \lambda$. Hence by the same reasoning as above there exists a $\delta > 0$ such that $v'(x) > 0$ for $x \in [t + \varepsilon_0, t + \varepsilon_0 + \delta)$ which is contrary to the definition of ε_0. Therefore $v(x) > 0$ and is increasing for $x > t$. But this is contrary to $v(b) = 0$. Thus, we have (1.53.b). Also, the same proof except for replacing $v''(t) > 0$ by $v''(t) \geq 0$ and $v'(t) \geq 0$ by $v'(t) > 0$ yields (1.53.c).

Now, suppose $u(t_1) > v(t_1)$ for some $t_1 \in [a,b)$. Arguing like the beginning of the proof of (i), we face situation 1° or 2° as in (i). Now, 1° is not possible as before. To handle 2°, i.e. there exists a $\beta < b$ such that $u(x) > v(x) > 0$ for $x \in [a,\beta)$ and $u(\beta) = v(\beta)$. Note that (1.53.a) and (1.53.c) imply

$$\text{if} \quad \tau(t) \in \left[0, \frac{\pi}{2}\right) \quad \text{then} \quad \frac{d\tau(t)}{dx} > 0 \ . \tag{1.54.a}$$

Hence, if $\theta \in \left[0, \frac{\pi}{2}\right)$ this implies $\tau(a) > \theta$. So 2° is contrary to (1.37). It remains to show the case of $\theta = \frac{\pi}{2}$. In this case, since $\theta = \frac{\pi}{2}$, if $\tau(a) > \frac{\pi}{2}$ by (1.37). $u(a) < v(a)$ which violates the assumption of 2°. Hence, $\tau(a) < \frac{\pi}{2}$ and $v'(a) > 0$. Let

$$s = \operatorname{Inf}\{\alpha \mid v'(x) > 0 \quad \text{for} \quad x \in (\alpha,a]\} \ .$$

Since $v'(a_1) = 0$, $s > a_1$. Clearly $v'(s) = 0$. If $v''(x) < 0$ for $x \in (s,a)$, $v'(a) < 0$ which is absurd. Therefore there exists an $\alpha_1 \in (s,a)$ such that $v''(\alpha_1) > 0$. By (1.1.a), $F_1(\alpha_1, v(\alpha_1)) > \lambda$. However, since by the definition of s, $v'(\alpha) > 0$ and $\tau(\alpha) \in \left[0, \frac{\pi}{2}\right)$ which is contrary to (1.53.b). Therefore 2° is also impossible when $\theta = \frac{\pi}{2}$. So we obtain (1.48) for $\theta \in \left[0, \frac{\pi}{2}\right]$ and consequently (1.49).

To prove (1.50) let $u = V_+(\lambda,a,\infty,0)$, $v = V_+(\lambda,a_1,\infty,0)$. We first note that (1.53.a) and (1.53.b) are still valid. The only difference in the proof of (1.53.b) is to replace $v(b) = 0$ by $\lim_{x\to\infty} v(x) = 0$. We omit the proof for the same reason as mentioned in (ii).

(iv) and (v) immediately follow from (1.46), (1.48) in (ii) and in (iii) respectively.

If (f.1) is satisfied and the equality occurs, by the uniqueness result for the initial value problem $u \equiv v$. In (i), this implies $v(b) = 0$. Since $v > 0$ in $[a,b_1]$, $v'(b) = 0$. But v cannot have a double zero. Thus the inequalities (1.46) and (1.47) must be strict. Similarly $u \equiv v$ cannot occur in (ii), (iv) and (v). To consider (iii). If $\theta \in \left(0, \frac{\pi}{2}\right)$, as we mentioned before by (1.54.a), $\tau(a) > \theta$. Hence, from Corollary 1.32, $u(a) < v(a)$. So $u \equiv v$ is not possible. The remaining case is $\theta = \frac{\pi}{2}$. Again, let $v = V_+(\lambda, a_1, b, \cdot)$. We claim

$$\text{if } b < +\infty \text{ and } \tau(t) = \frac{\pi}{2} \text{ for } t \in [a_1, b)$$

$$\text{then } F_1(\lambda, v(t) < \lambda . \tag{1.54.b}$$

Indeed, pick a $b_1 > b$ and put $v_1 = V_+(\lambda, t, b_1, \cdot)$. By (i), we have $v_1(t) > v(t)$. Thus, if $F_1(\lambda, v(t)) > \lambda$, by (F.4), $F_1(\lambda, v_1(t)) > \lambda$. This together with $v_1'(t) = 0$ contradicts (1.53.b). Now if $u = V_+(\lambda, a, b, \cdot)$, for $b < +\infty$, and $u \equiv v$, then $v'(a_1) = v'(a) = 0$. By (1.54.b) and (1.1.a), $v''(a_1) < 0$ and $v''(a) < 0$. Hence, there exists an $\varepsilon > 0$ such that $v'(x) < 0$ for $x \in (a_1, a_1 + \varepsilon)$ and $v'(x) > 0$ for $x \in (a - \varepsilon, a)$. This implies v has a minimum at some point $\beta \in (a_1, a)$. Thus $v'(\beta) = 0$ and $v''(\beta) > 0$ i.e. $\tau(\beta) = \frac{\pi}{2}$ and $F_1(\beta, v(\beta)) > \lambda$ which is contrary to (1.54.b). Therefore inequalities (1.48) and (1.49) must be strict even for the case $\theta = \frac{\pi}{2}$.

To show the last assertion, let $u = V_+(\lambda, a, \infty, \cdot)$ and $v = V_+(\lambda, a_1, \infty, \cdot)$. Suppose $u \equiv v$ then $v'(a) = v'(a_1) = 0$. Obviously, if $v''(a) > 0$, by (1.1.a) $F_1(a, v(a)) > \lambda$ which is contrary to (1.53.b). Suppose $v''(a) < 0$, then $v'(x) > 0$ for $x \in (a - \varepsilon, a)$ with some $\varepsilon > 0$. Let

$$z = \operatorname{Inf}\{\alpha \mid v'(x) > 0 \text{ for } x \in (\alpha, a)\} .$$

Obviously, $z > a_1$. On the other hand, $v(x) < v(a)$ for $x \in [\alpha, a)$. Since $\frac{\partial F_1}{\partial x} > 0$ and by (F.4), $\frac{\partial F_1}{\partial y} > 0$, we have $F_1(x,v(x)) < F_1(a,v(a))$ for $x \in [\alpha,a)$. By (1.1.a), $F_1(a,v(a)) < \lambda$. Hence, for $x \in [\alpha,a)$, $F_1(x,v(x)) < \lambda$ and by (1.1.a), $v''(x) < 0$. This together with $v'(a) = 0$ implies $v'(z) > 0$ which contradicts the definition of z. Thus $v''(a) < 0$ is not possible and it remains that $v''(a) = 0$. By (1.1.a), $F_1(a,v(a)) = \lambda$. Define the function L by $F_1(x,L(x)) = \lambda$. Clearly, the assumption $\frac{\partial F_1}{\partial y} > 0$ implies $L(x)$ is well-defined. It is also easy to check by implicit function theorem that $L'(x) = -\frac{\partial F_1}{\partial x}(x,L(x))/\frac{\partial F_1}{\partial y}(x,L(x))$. Since $L'(a) < 0 = v'(a)$ and $L(a) = v(a)$, $L(x) > v(x)$ for $x \in (a-\varepsilon,a)$ with some $\varepsilon > 0$. This implies $F_1(x,v(x)) < \lambda$ and by (1.1.a), $v''(x) < 0$ for such x. Hence $v'(x) < 0$ for those x. Then arguing like the case $v''(a) < 0$, we obtain a contradiction again. This completes the proof.

Remark 1.55

(a) (1.53.c) and (1.1.a) imply $V_+''(\lambda,a,b,\theta,a) < 0$ for $b < +\infty$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ provided that (r.1), (F.1), (F.3) and (F.4) are satisfied and $F_1(x,y)$ is nondecreasing in x. Also note that the proof of (1.53.c) does not need the assumption (F.2). However, it ensures the existence of $V_+(\lambda,a,b,\theta,\cdot)$.

(b) If (r.1), (F.1)-(F.4) and (f.1) are satisfied (1.54.b) and (1.1.a) imply $V_+''(\lambda,a,b,\frac{\pi}{2},a) < 0$ for $b < +\infty$. Moreover, (F.2) can be replaced by an assumption which ensures the existence of $V_+(\lambda,a,b_1,\frac{\pi}{2},\cdot)$ for some $b_1 \in (b,\infty)$.

From Corollaries 1.32 and 1.45, we have an immediate consequence:

Corollary 1.56

Assume (r.1), (F.1)-(F.4) are satisfied. Let $\lambda > 0$, $a > 0$ and $\theta \in \left[0, \frac{\pi}{2}\right]$ be fixed. If u is a solution of $(I)_a$, then $V_-(\lambda,a,\infty,\theta,x) \leq u(x) \leq V_+(\lambda,a,\infty,\theta,x)$ for $x \in [a,\infty)$.

Proof

If $u \equiv 0$, there is nothing to prove. Suppose $u \not\equiv 0$. Let z_i be i-th zero of u in (a,∞), $i = 1,2,\ldots$. Note that if u has infinitely many zeroes in (a,∞), (1.41.c) together with Proposition 1.43 (i) shows that $\{z_i\}$ cannot have an accumulated point in (a,∞). Since on each interval $[z_i,z_{i+1}]$, $u(x) = V_+(\lambda,z_i,z_{i+1},0,x)$ or $V_-(\lambda,z_i,z_{i+1},0,x)$. Also, on the interval $[a,z_1]$, $u(x) = V_+(\lambda,a,z_1,\theta,x)$ or $V_-(\lambda,a,z_1,\theta,x)$. By Corollary 1.45 we know

$$V_-(\lambda,a,\infty,0,x) \leq V_\pm(\lambda,z_i,z_{i+1},0,x) \leq V_+(\lambda,a,\infty,0,x), \quad x \in [z_1,z_2]$$

and

$$V_-(\lambda,a,\infty,\theta,x) \leq V_\pm(\lambda,a,z_1,\theta,x) \leq V_+(\lambda,a,\infty,\theta,x), \quad x \in [a,z_1] .$$

Since, by Corollary 1.32,

$$V_-(\lambda,a,\infty,\theta,\cdot) \leq V_\pm(\lambda,a,\infty,0,\cdot) \leq V_+(\lambda,a,\infty,\theta,\cdot)$$

for $\theta \in \left[0, \frac{\pi}{2}\right]$, we complete the proof if u has infinitely many zeroes. Suppose u has only finitely many zeroes, say ℓ. Then, on the interval $[z_\ell,\infty)$, $u(x) = V_+(\lambda,z_\ell,\infty,0,x)$ or $V_-(\lambda,z_\ell,\infty,0,x)$. By Corollary 1.45,

$$V_-(\lambda,a,\infty,0,x) \leq V_\pm(\lambda,z_\ell,\infty,0,x) \leq V_+(\lambda,a,\infty,0,x), \quad x \in [z_\ell,\infty) .$$

Thus, together with the above inequalities, the result follows.

Having established these uniqueness and "monotonicity" properties, we are going to obtain several upper bounds for u and u' in terms of λ, r_2 and bounds involving ω_1 and ω_2.

Lemma 1.57

Suppose (r.1), (F.1) and (F.2) are satisfied. Let $\lambda > 0$ and $0 < \theta < \frac{\pi}{2}$ be fixed. Let u be a solution of $(I)_{a,b}$. Then, for any $0 < a < b < +\infty$,

$$\|u\|_{L^2[a,b]} < \tilde{K}_1(\lambda,a,b) < K_1(\lambda,a) \tag{1.58.a}$$

$$\|u'\|_{L^2[a,b]} < \tilde{K}_2(\lambda,a,b) < K_2(\lambda,a) \tag{1.58.b}$$

$$\|u\|_{L^\infty[a,b]} < \tilde{K}_3(\lambda,a,b) < K_3(\lambda,a) \tag{1.58.c}$$

$$\|u'\|_{L^\infty[a,b]} < \tilde{K}_4(\lambda,a,b) < K_4(\lambda,a) \tag{1.58.d}$$

where

$$\tilde{K}_1(\lambda,a,b) = \sum_{i=1}^{2} (\lambda r_2)^{1/\sigma_i} \left(\int_a^b \omega_i^{-2/\sigma_i} dx\right)^{1/2}$$

$$\tilde{K}_2(\lambda,a,b) = (\lambda r_2)^{1/2} \tilde{K}_1$$

$$\tilde{K}_3(\lambda,a,b) = (2\tilde{K}_1 \cdot \tilde{K}_2)^{1/2}$$

$$\tilde{K}_4(\lambda,a,b) = (\lambda r_2)^{1/2} \tilde{K}_3$$

and

$$K_i(\lambda,a) = \lim_{b\to\infty} \tilde{K}_i(\lambda,a,b), \qquad 1 < i < 4 . \tag{1.59}$$

Proof

Suppose first u is a positive solution of $(I)_{a,b}$. Multiplying (1.4.a) by u and integrating it by parts we have

$$u'(a)u(a) - u'(b)u(b) + \int_a^b u'^2dx + \int_a^b F(x,u)u^2dx = \lambda \int_a^b r(x)u^2dx .$$

In view of the boundary conditions in (1.4.b), we know $u(b) = 0$ and

$$u'(a)u(a) > 0 .$$

Hence

$$\int_a^b u'^2dx + \int_a^b F(x,u)u^2dx < \lambda \int_a^b r(x)u^2dx . \tag{1.60}$$

By assumptions (F.2) and (r.1), this leads to

$$\int_a^b \omega_1|u|^{\sigma_1+2}dx < \lambda r_2 \int_a^b u^2dx . \tag{1.61}$$

Next, writing u^2 as the product of $\omega_1^{-2/(\sigma_1+2)}$ and $(\omega_1|u|^{\sigma_1+2})^{2/(\sigma_1+2)}$ and applying Hölder's inequality, we obtain

$$\int_a^b u^2 < c^{\sigma_1/(\sigma_1+2)}(\int_a^b \omega_1|u|^{\sigma_1+2})^{2/(\sigma_1+2)} \tag{1.62}$$

where $c = \int_a^b \omega_1^{-2/\sigma_1}dx$. Combining this with (1.61) yields

$$\int_a^b u^2dx < c^{\sigma_1/(\sigma_1+2)}(\lambda r_2 \int_a^b u^2dx)^{2/(\sigma_1+2)} .$$

Consequently, we get

$$\int_a^b u^2dx < (\lambda r_2)^{2/\sigma_1} \int_a^b \omega_1^{-2/\sigma_1}dx . \tag{1.63}$$

Going back to (1.60) and applying the assumption (r.1) again we have

$$\int_a^b u'^2dx < \lambda r_2 \int_a^b u^2dx .$$

Thus, with the help of (1.63), this implies

$$\int_a^b u'^2 dx < (\lambda r_2)^{(2+\sigma_1)/\sigma_1} \int_a^b \omega_1^{-2/\sigma_1} dx .$$

Likewise, the same proof shows that if u is a negative solution of $(I)_{a,b}$ then

$$\int_a^b u^2 dx < (\lambda r_2)^{2/\sigma_2} \int_a^b \omega_2^{-2/\sigma_2} dx$$

and

$$\int_a^b u'^2 dx < (\lambda r_2)^{(2+\sigma_2)/\sigma_2} \int_a^b \omega_2^{-2/\sigma_2} dx .$$

To obtain estimates (1.58.a) and (1.58.b) for solutions in an arbitrary nodal class, we assume u has interior zeroes at $z_1, z_2, \ldots, z_m$, $m > 1$. The restriction of u to each interval of $[a,z_1],[z_1,z_2],\ldots,[z_m,b]$ is either positive or negative. Hence (1.58.a) and (1.58.b) hold for u on each interval with the corresponding end-points. Summing up these estimates completes the proof of (1.58.a) and (1.58.b).

To prove (1.58.c), it is easy to see that

$$u^2(x) = -\int_x^b 2uu'dt$$

for $x \in [a,b]$. Applying Schwartz inequality, we obtain

$$u^2(x) < 2\left(\int_x^b u^2 dt\right)^{1/2}\left(\int_x^b u'^2 dt\right)^{1/2}$$

$$< 2\left(\int_a^b u^2 dt\right)^{1/2}\left(\int_a^b u'^2 dt\right)^{1/2} .$$

This together with (1.58.a) and (1.58.b) leads to (1.58.c).

Finally, we need the following lemma to prove (1.58.d).

Lemma 1.64

Suppose (r.1) and (F.1) are satisfied. Let $\lambda > 0$ and u be a solution of $(I)_{a,b}$ (resp. $(I)_a$) then

$$\|u'\|_{L^\infty[a,b]} < \sqrt{\lambda r_2}\,\|u\|_{L^\infty[a,b]} \quad \left(\text{resp. } \|u'\|_{L^\infty[a,\infty)} < \sqrt{\lambda r_2}\,\|u\|_{L^\infty[a,\infty)}\right) . \tag{1.65}$$

Assuming the lemma, (1.58.d) easily follows from (1.58.c). Now, we prove the lemma.

Proof

If $u \equiv 0$, (1.65) is clearly satisfied. To consider nontrivial solutions we put $R(x) = \lambda(r_2 + \delta)u^2(x) + u'^2(x)$ with $\delta > 0$. Then it follows, with the aid of equation (1.1.a) that

$$R'(x) = 2[F(x,u(x)) + \lambda(r_2 + \delta - r(x))]u(x)u'(x) .$$

From the assumption (F.1), we know $F(x,u(x)) > 0$. Also note that $r_2 + \delta - r(x) > 0$. Thus $R'(x)$ has the same sign as that of $u(x)u'(x)$.

Since it is easy to see that if $u(t) = 0$ there exists an $\varepsilon > 0$ such that

$$u(x)u'(x) > 0 \quad \text{for} \quad t < x < t + \varepsilon ,$$
$$u(x)u'(x) < 0 \quad \text{for} \quad t - \varepsilon < x < t$$

and hence

$$R'(x) > 0 \quad \text{for} \quad t < x < t + \varepsilon ,$$
$$R'(x) < 0 \quad \text{for} \quad t - \varepsilon < x < t .$$

Therefore R cannot attain its maximum at interior zeroes of u as well as at b in the case of $(I)_{a,b}$.

At the point a, if $\theta = 0$ i.e. $u(a) = 0$ the same reasoning as above shows $R(a)$ is not a maximum of $R(x)$ either. Next, for $0 < \theta < \frac{\pi}{2}$, $u'(a)$ and $u(a)$ have the same sign. Thus, $R'(a) > 0$ which again implies $R(a)$ is not a maximum of $R(x)$. Finally, in the case of $(I)_a$, it follows from Lemma 1.7 that

$$\lim_{x\to\infty} R(x) = 0 .$$

However, $R(a) = \lambda(r_2 + \delta)u^2(a) + u'^2(a) > 0$. Thus in all cases ($(I)_{a,b}$ and $(I)_a$, all fixed θ, $0 < \theta < \frac{\pi}{2}$) we conclude that R must attain its maximum at a point t at which $R'(t) = 0$ and $u'(t) = 0$. Therefore

$$\|u'\|_{L^\infty} < \|\sqrt{R}\|_{L^\infty} = \sqrt{R(t)} = \sqrt{\lambda(r_2+\delta)}\,|u(t)| < \sqrt{\lambda(r_2+\delta)}\,\|u\|_{L^\infty} . \qquad (1.66)$$

Since (1.66) holds for every $\delta > 0$, we get (1.65).

Proof of Theorem 1.2

Let $b_n = a + n$ and $u_n = V_+(\lambda,a,b_n,\theta,\cdot)$, $n = 1,2,3,\ldots$. Put $c = K_3(\lambda,a) + K_4(\lambda,a)$. Then, for $n > 1$, Lemma 1.57 implies

$$\|u_n\|_{C^1[a,b_n]} < c . \qquad (1.67)$$

Let $c_1(n) = \max_{\substack{x\in[a,b_n] \\ y\in[0,c]}} |f(x,y)|$, it follows from equation (1.1.a) that

$$\|u_\ell''\|_{L^\infty[a,b_n]} < \lambda r_2 c + c_1(n) \qquad (1.68)$$

for all $\ell > n$.

The bounds (1.67) and (1.68), the Arzela-Ascoli theorem and (1.1.a) imply that there exists a subsequence $\{u_{n_k}\}$ and a $u \in c^2[a,\infty)$ such that

$$u_{n_k} \xrightarrow{c^2} u \text{ uniformly on compact subsets of } [a,\infty) . \qquad (1.69)$$

Also, note that $u_n(a)\cos\theta - u_n'(a)\sin\theta = 0$ for all n. This implies $u(a)\cos\theta - u'(a)\sin\theta = 0$. The "monotonicity" result (1.46) tells us $u_{n+1}(x) > u_n(x)$ for $x \in [a,b_n]$. Hence $u(x) > 0$ for $x \in (a,\infty)$.

To show $u \in H^1[a,\infty)$ put

$$v_k(x) = \begin{cases} u_{n_k}(x) & \text{if } a < x < b_{n_k} \\ 0 & \text{if } x > b_{n_k} . \end{cases}$$

Pick an $M > 0$, by Lemma 1.57, we have

$$\int_a^M v_k^2 + v_k'^2 dx < K_1^2 + K_2^2$$

for all k. Invoking (1.69) we get

$$\int_a^M u^2 + u'^2 dx < K_1^2 + K_2^2 . \qquad (1.70)$$

Since (1.70) is true for all $M > a$, we conclude $u \in H^1[a,\infty)$.

<u>Remark 1.71</u>

The "monotonicity" result (1.46) indicates that not only a subsequence $\{u_{\ell_k}\}$ but the whole sequence $\{u_\ell\}$ converges to u.

<u>Corollary 1.72</u>

Assume (r.1), (F.1) and (F.2) are satisfied. If u is a solution of $(I)_a$ then

$$\|u\|_{L^2[a,\infty)} < K_1(\lambda,a) , \tag{1.73.a}$$

$$\|u'\|_{L^2[a,\infty)} < K_2(\lambda,a) , \tag{1.73.b}$$

$$\|u\|_{L^\infty[a,\infty)} < K_3(\lambda,a) , \tag{1.73.c}$$

$$\|u'\|_{L^\infty[a,\infty)} < K_4(\lambda,a) . \tag{1.73.d}$$

Proof

From Lemma 1.7 we know

$$\lim_{x\to\infty} u(x) = 0$$

and

$$\lim_{x\to\infty} u'(x) = 0 .$$

With this replacing the boundary condition $u(b) = 0$, the rest of the proof can be easily carried out by the same argument as in Lemma 1.57. We omit it.

Remark 1.74

(a) In the proof of Corollary 1.72, the solution u may have infinitely many zeroes in $[a,\infty)$. However, this does not affect in the proof and implies that any solution of equation (1.1.a) satisfying the boundary conditions $u(a)\cos\theta - u'(a)\sin\theta = 0$, $0 < \theta < \frac{\pi}{2}$, and having infinitely many zeroes, automatically belongs $L^2[a,\infty)$, provided that (r.1), (F.1), (F.2) are assumed.

(b) An example of solutions having infinitely many zeroes was given by Heinz [8], where he also gave a sufficient condition which prohibits the existence of such solutions.

(c) $K_i(\lambda,a)$, $1 < i < 4$, are continuous functions of a and λ.

For fixed $a > 0$

$$\lim_{\lambda \to 0} K_i(\lambda,a) = 0 \,. \tag{1.75.a}$$

If $\lambda > 0$ is fixed, then

$$\lim_{a \to \infty} K_i(\lambda,a) = 0 \,. \tag{1.75.b}$$

Before completing this section we are going to discuss the continuous dependence of positive and negative solutions on parameters and domains.

Proposition 1.76

Assume (r.1), (F.1), (F.2), (F.3) and (F.4) are satisfied. Let $0 < \theta < \frac{\pi}{2}$ be fixed. Let $\lambda > 0$, $0 < a < b < +\infty$ and $\{(\lambda_k,a_k,b_k)\}$ be a sequence such that $\lim_{k \to \infty} (\lambda_k,a_k,b_k) = (\lambda,a,b)$, where b_k could be $+\infty$ if $b = +\infty$.

(i) If $b < +\infty$ and $\lambda < \mu_1(a,b)$ then

$$\lim_{k \to \infty} u_\pm(\lambda_k,a_k,b_k,x) = 0$$

and

$$\lim_{k \to \infty} u_\pm'(\lambda_k,a_k,b_k,x) = 0$$

uniformly for $x \in [a,b]$.

(ii) If either $b < +\infty$ and $\lambda > \mu_1(a,b)$ or $b = +\infty$ then

$$\lim_{k \to \infty} u_\pm(\lambda_k,a_k,b_k,x) = u_\pm(\lambda,a,b,x)$$

and

$$\lim_{k \to \infty} u_\pm'(\lambda_k,a_k,b_k,x) = u_\pm'(\lambda,a,b,x)$$

for $x \in [a,b]$ or $[a,\infty)$ in case that $b = +\infty$ and uniformly on compact subsets of (a,b).

<u>Proof</u>

We only need to prove uniform convergence on compact subsets by which, together with the C^2-smoothness of the solutions, the convergence at the end points follows.

Let $v_k = V_+(\lambda_k, a_k, b_k, \cdot)$. Then arguments analogous to those of Theorem 1.2 show that by passing to a subsequence if necessary, there is a function $v(x)$

$$v_k \xrightarrow{C^1} v \text{ uniformly on compact subsets of } (a,b)$$

and $v(x)$ is a solution of $(I)_{a,b}$. Hence, by Proposition 1.43 (i) v must be the trivial solution. Since every subsequence of $\{v_k\}$ does so. We complete the proof of (i).

To prove (ii). Let v_k be defined as above and the same argument gives $v(x)$. Since $v(x) > 0$ by Theorem 1.5, we complete (ii) provided that v is not the trivial solution.

To show that v cannot be the trivial solution, we first treat the case $b < +\infty$, $\lambda > \mu_1(a,b)$. Suppose v is the trivial solution then

$$\|v_k\|_{C^1[\alpha,\beta]} \to 0 \text{ for any compact subsets } [\alpha,\beta] \text{ of } (a,b) . \quad (1.77)$$

Put $y_k = \dfrac{v_k}{\|v_k\|}$ where $\|\cdot\| = \|\cdot\|_{C^1[a_k,b_k]}$ then $\|y_k\|_{C^1[a_k,b_k]} = 1$.

Thus the same argument as above shows that by passing to a subsequence if necessary, there is a function $y(x)$ such that $\|y\|_{C^1[a,b]} = 1$ and

$$y_k \xrightarrow{c^1} y \text{ uniformly on compact subsets of } (a,b) . \tag{1.78}$$

From (1.77), (1.78), (1.4) and the assumption (F.3), we know that $(\lambda, y(x))$ is a solution of (1.40). Since $y \geq 0$ and $\|y\|_{c'[a,b]} = 1$, $y(x) > 0$ for $x \in (a,b)$. Thus $\lambda = \mu_1(a,b)$ which is contrary to the assumption $\lambda > \mu_1(a,b)$.

Next, in case that $b = +\infty$, we can pick a $\beta_1 < +\infty$ such that, from (1.41.d), $\lambda > \mu_1(a,\beta_1)$. By (1.46) and the result we proved above, v cannot be the trivial solution.

§2. EXISTENCE OF SOLUTIONS WITH A PRESCRIBED NUMBER OF NODES WHEN THE NONLINEARITY IS ODD

Now we turn to the questions of solutions with nodes. Let $\lambda > 0$, $a > 0$, $0 < \theta < \frac{\pi}{2}$ and $n > 1$ be an integer. Let $S^{+}_{a,n}(\lambda,\theta)$ (resp. $S^{-}_{a,n}(\lambda,\theta)$) denote the set of $u \in C^2[a,\infty) \cap H^1[a,\infty)$ such that u satisfies (1.1.a) and the boundary condition $u(a)\cos\theta - u'(a)\sin\theta = 0$, $u > 0$ (resp. < 0) in a deleted neighborhood of $x = a$, u has exactly $n - 1$ simple zeroes in (a,∞). We will show the existence of solutions in each nodal class $S^{\pm}_{a,n}(\lambda,\theta)$. We can now state the main result of this section.

Theorem 2.1

Assume (r.1), (F.1)-(F.4) and

(F.5) $F(x,-y) = F(x,y)$ for $x \in [0,\infty)$, $y \in \mathbb{R}$

are satisfied. Let $\lambda > 0$, $a > 0$ and $0 < \theta < \frac{\pi}{2}$ be given, then $S^{+}_{a,n}(\lambda,\theta)$ and $S^{-}_{a,n}(\theta)$ are nonempty for all $n \in \mathbb{N}$.

Remark 2.2

The existence of nodal solutions has been obtained in [4] and [6]-[8]. However, we generalize the result in several directions as mentioned in Remark 1.3.

To prove the theorem, we will generalize a result of Hempel for bounded intervals (Proposition 2.4, also see [17] or [18]) to the unbounded case and use it to find solutions with a prescribed number of nodes when $\theta = 0$. Starting from a solution belonging to $S^{\pm}_{a,n}(\lambda,0)$, those "monotonicity" properties, which were developed in the previous section, allow us to set up an iteration scheme to construct a solution with $n - 1$ nodes in the case $0 < \theta < \frac{\pi}{2}$.

Let us assume (r.1), (F.1), (F.3) and (F.4). Also, for $0 < a < b < +\infty$, we assume $V_+(\lambda,a,b,0,\cdot)$ (resp. $V_-(\lambda,a,b,0,\cdot)$) exists whenever $\lambda > \mu_1(a,b,0)$. Define the number $\Lambda^+[a,b]$ (resp. $\Lambda^-[a,b]$) by

$$\Lambda^+[a,b] \text{ (resp. } \Lambda^-[a,b]) = \int_a^b [\lambda r(x)u^2(x) - (u'(x))^2 - 2\int_0^{u(x)} f(x,y)dy]dx \tag{2.3.a}$$

where

$$u = \begin{cases} 0 & \text{if } \lambda \le \mu_1(a,b,0) \\ V_+(\lambda,a,b,0,\cdot) \text{ (resp. } V_-) & \text{if } \lambda > \mu_1(a,b,0) . \end{cases} \tag{2.3.b}$$

To make the notation clear, let us recall that $V_\pm$ and μ_1 were defined as in Remark 1.6 and in (1.41) respectively. From Theorem 1.5, we know $\Lambda^\pm[a,b]$ are well-defined. If $F(x,y)$ satisfies (F.5), then $\Lambda^+[a,b] = \Lambda^-[a,b]$ due to the fact that $V_+ = -V_-$, and we simply use the notation $\Lambda[a,b]$. Also, for convenience, we adapt the notation $V_\pm(\lambda,a,b,0,x) \equiv 0$ whenever $\lambda \le \mu_1(a,b,0)$.

<u>Proposition 2.4</u> (Hempel [17])

Assume (r.1), (F.1), (F.3), (F.4) and (F.5) are satisfied.

Suppose $u = V_+(\lambda,a,b,0,\cdot)$ exists if $\lambda > \mu_1(a,b,0)$. Then $\Lambda[a,b]$ is a differentiable function of a and b with derivatives given by

$$\frac{\partial\Lambda}{\partial a} = -(u'(a))^2 \tag{2.5.a}$$

and

$$\frac{\partial\Lambda}{\partial b} = (u'(b))^2 \; . \tag{2.5.b}$$

Remark 2.6

Hemple actually imposed the stronger assumption

(F.4)' There exists a $\varepsilon > 0$ such that for fixed $x > 0$, $y^{-\varepsilon}F(x,y)$ is a nondecreasing function of y if $y > 0$ and a nonincreasing function of y if $y < 0$

instead of (F.4). However, in view of his proof, (F.4) would be sufficient provided that $V_+(\lambda,a,b,0,\cdot)$ exists whenever $\lambda > \mu_1(a,b,0)$. Also, it is worthwhile to mention that (F.4)' insures the existence of V_+ due to Proposition 1.43 and Remark 1.44.

Corollary 2.7

Assume (r.1), (F.1), (F.3), (F.4) are satisfied. Suppose $V_+ = V_+(\lambda,a,b,0,\cdot)$ (resp. V_-) exists if $\lambda > \mu_1(a,b,0)$. Then $\Lambda^+[a,b]$ (resp. $\Lambda^-[a,b]$) is a differentiable function of a and b, with derivatives given by

$$\frac{\partial\Lambda^+}{\partial a} = -(V_+'(a))^2 \quad \left(\text{resp. } \frac{\partial\Lambda^-}{\partial a} = -(V_-'(a)^2\right) \tag{2.8.a}$$

and

$$\frac{\partial\Lambda^-}{\partial b} = (V_+'(b))^2 \quad \left(\text{resp. } \frac{\partial\Lambda^-}{\partial b} = (V_-'(b))^2\right) \; . \tag{2.8.b}$$

Proof

Since their proofs are the same only the first proof will be carried out. Let

$$H(x,y) = \begin{cases} F(x,y) & \text{if } y > 0 \\ F(x,-y) & \text{if } y < 0 . \end{cases}$$

Then $V_+(\lambda,a,b,0,\cdot)$ is also the positive solution of

$$-u'' = (\lambda r(x) - H(x,u))u$$

$$u(a) = u(b) = 0$$

and hence the result easily follows from Proposition 2.4.

To generalize Hempel's result to the case of an unbounded interval, we assume (r.1), (F.1), (F.2), (F.3) and (F.4) and define $\Lambda^+[a,\infty]$ (resp. $\Lambda^-[a,\infty]$) by

$$\Lambda^+[a,\infty] \quad (\text{resp. } \Lambda^-[a,\infty])$$

$$= \int_a^\infty [\lambda r(x)u^2(x) - (u'(x))^2 - 2\int_0^{u(x)} f(x,y)dy]dx \tag{2.9}$$

where $u = V_+(\lambda,a,b,0,\cdot)$ (resp. V_-) and again we simply use the notation $\Lambda[a,\infty]$ whenever (F.5) is satisfied. To justify the (2.9) is well-defined we show

<u>Proposition 2.10</u>

Suppose (r.1), (F.1)-(F.4) are satisfied then

$$0 < \Lambda^{\pm}[a,\infty] < +\infty .$$

To prove the proposition, we need a few lemmas.

<u>Lemma 2.11</u>

Suppose (r.1), (F.1)-(F.4) are satisfied. Let $\lambda > 0$ and $a > 0$ be fixed then $\Lambda^{\pm}[a,b]$ are nondecreasing functions of b. Moreover

(i) $\Lambda^{\pm}[a,b] = 0$ if $\lambda < \mu_1(a,b,0)$ and

(ii) $\Lambda^{\pm}[a,b] > 0$ if $\lambda > \mu_1(a,b,0)$.

Proof

From (2.3.b), $\Lambda^{\pm}[a,b] = 0$ if $\lambda < \mu_1(a,b)$. Applying Corollary 2.7, we complete the proof.

Lemma 2.12

Suppose (r.1), (F.1), (F.3) and (F.4) are satisfied. Let u be a solution of $(I)_a$ then

$$\int_0^\infty \int_0^{u(x)} f(x,y)dydx < +\infty .$$

Proof

Multiplying equation (1.1.a) by u and integrating by parts, we obtain

$$\int_a^b \lambda r(x)u^2dx - \int_a^b f(x,u)udx = u'(a)u(a) - u'(b)u(b) + \int_a^b u'^2dx .$$

Since $0 < \theta < \frac{\pi}{2}$ it follows from (1.1.b) that $u'(a)u(a) > 0$ and hence

$$\int_a^b f(x,u)udx < \int_a^b \lambda r(x)u^2dx - \int_a^b u'^2dx + u'(b)u(b) .$$

By Lemma 1.7, $u \in H^1[a,\infty)$ and $u'(b)u(b) \to 0$ as $b \to +\infty$. Thus, there exists a constant C (independent of b) such that

$$\int_a^b f(x,u)udx < C .$$

Letting $b \to \infty$, we get

$$\int_a^\infty f(x,u(x))u(x)dx < C .$$

By the assumption (F.4), $f(x,y)$ is increasing in y if $y > 0$ and decreasing in y if $y < 0$. Hence

$$\int_a^\infty \int_0^{u(x)} f(x,y)\,dy\,dx < \int_a^\infty f(x,u(x))u(x)\,dx < C \ .$$

Lemma 2.13

Assume (r.1), (F.1)-(F.4) are satisfied, then

$$\lim_{b\to\infty} \Lambda^{\pm}[a,b] = \Lambda^{\pm}[a,\infty] \ .$$

Proof

Let $u = V_+(\lambda,a,\infty,0,\cdot)$. From the proof of Theorem 1.2 and Remark 1.71, any sequence $\{u_k\}$ with $u_k = V_+(\lambda,a,b_k,0,\cdot)$ and $b_k \to +\infty$ has the property that

$$u_k \xrightarrow{C^1} u \text{ uniformly on compact subintervals of } [a,\infty) \ . \tag{2.14}$$

Multiplying (1.1.a) by u_k and integrating by parts, we get

$$\int_x^{b_k} u_k'^2\,dt + \int_x^{b_k} F(t,u_k)u_k\,dt = u_k(b_k)u_k'(b_k)$$

$$- u_k(x)u_k'(x) + \lambda \int_a^b r(t)u_k^2\,dt \ .$$

Since $u_k(b_k) = 0$ and $F > 0$ this leads to

$$\int_x^{b_k} u_k'^2\,dt < - u_k(x)u_k'(x) + \lambda \int_x^{b_k} r(t)u_k^2\,dt \ . \tag{2.15}$$

From (1.46), we know for $x \in [a,b_k]$ that

$$u_k(x) < u(x) \ .$$

By Lemma 1.57

$$\|u_k'\|_{L^\infty[a,b_k]} < K_4(\lambda,a) \ .$$

Since $\lim_{x\to\infty} u(x) = 0$. Given $\varepsilon > 0$, there exists an $s > a$ such that if $x > s$

$$|u_k(x)u_k'(x)| < \varepsilon \tag{2.16}$$

uniformly in k. Also, note that (1.46) and (r.1) imply

$$\int_x^{b_k} \lambda r(t)u_k^2 dt \leq \lambda r_2 \int_x^{\infty} u^2 dt .$$

Since $u \in H^1[a,\infty)$ we have, for large x, that

$$\int_x^{b_k} \lambda r(t)u_k^2 dt < \varepsilon . \tag{2.17}$$

Since (2.16) and (2.17) hold uniformly in k, by (2.15),

$$\int_x^{b_k} u_k'^2 dt < 2\varepsilon \tag{2.18}$$

uniformly in k for large x. Hence (2.14), (2.17) and (2.18) imply

$$\lim_{k\to\infty} \int_a^{b_k} \lambda r(t)u_k^2 dt = \int_a^{\infty} \lambda r(t)u^2 dt$$

and

$$\lim_{k\to\infty} \int_a^{b_k} u_k'^2 dt = \int_a^{\infty} u'^2 dt .$$

Thus it remains to prove

$$\lim_{k\to\infty} \int_a^{b_k} \int_0^{u_k(t)} f(t,y)dydt = \int_a^{\infty} \int_0^{u(t)} f(t,y)dydt . \tag{2.19}$$

From (1.46) and (F.4) we know

$$\int_x^{b_k} \int_0^{u_k(t)} f(t,y)dydt \leq \int_x^{\infty} \int_0^{u(t)} f(t,y)dydt .$$

Given $\varepsilon > 0$, by Lemma 2.12, there exists an $s_1 > a$ such that if

$x > s_1$

$$\int_x^\infty \int_0^{u(t)} f(t,y)\,dy\,dt < \varepsilon .$$

Hence, if $x > s_1$

$$\int_x^{b_k} \int_0^{u_k(t)} f(t,y)\,dy\,dt < \varepsilon$$

uniform in k. Since (2.14) implies

$$\left|\int_a^{s_1} \int_0^{u(t)} f(t,y)\,dy\,dt - \int_a^{s_1} \int_0^{u_k(t)} f(t,y)\,dy\,dt\right| < \varepsilon$$

for large k. Therefore (2.19) follows from the standard 3ε argument.

Proof of Proposition 2.10

From (1.41.d), we can pick a $b > a$ such that $\lambda > \mu_1(a,b,0)$ and hence, by Lemma 2.11, $\Lambda^{\pm}[a,b] > 0$ and $\Lambda^{\pm}[a,b_1] > 0$ for all $b_1 > b$. Therefore, by Lemma 2.13, $\Lambda^{\pm}[a,\infty] > 0$.

The assertion $\Lambda^{\pm}[a,\infty] < +\infty$ follows from (2.9). Theorem 1.2 and Lemma 2.12.

Now, we have an analogue of Hempel's result.

Proposition 2.20

Suppose (r.1), (F.1)-(F.4) are satisfied. Then $\Lambda^{+}[a,\infty]$ (resp. $\Lambda^{-}[a,\infty]$) is a differentiable function of a and

$$\frac{d\Lambda^{+}}{da} \quad (\text{resp. } \frac{d\Lambda^{-}}{da}) = -(u'(a))^2 \tag{2.21}$$

where $u = V_{+}(\lambda,a,\infty,0,\cdot)$ (resp. V_{-}).

Proof

Since their proofs are the same, only the case $\Lambda^+[a,\infty]$ will be carried out. Let $\{b_n\} \subset (a,\infty)$ be an increasing sequence such that $\lim_{n\to\infty} b_n = +\infty$ and put

$$\varphi_n(x) = \Lambda^+[x,b_n] \quad \text{for} \quad x \in [0,b_1] \ .$$

Clearly, by (2.8.a),

$$\varphi_n'(x) = -(V_+'(\lambda,x,b_n,0,x))^2$$

and from Proposition 1.76, φ_n' are continuous on $[0,b_1]$. It follows from (1.47) that

$$\varphi_n'(x) > \varphi_{n+1}'(x) \tag{2.22.a}$$

for $n = 1,2,3,\ldots$ and for every $x \in [0,b_1]$. Let $\psi(x) = -(V_+'(\lambda,x,\infty,0,x))^2$. By Proposition 1.76, ψ is continuous and

$$\lim_{n\to\infty} \varphi_n'(x) = \psi(x) \ . \tag{2.22.b}$$

It follows from Dini Theorem ([26], Chap. 7), with the aid of (2.22.a) and (2.22.b), that

$$\varphi_n' \to \psi \quad \text{uniformly on} \quad [0,b_1] \ . \tag{2.22.c}$$

Put $\varphi(x) = \Lambda^+[x,\infty]$. Then an elementary theorem in Calculus ([26], Chap. 7) together with Lemma 2.13 and (2.22.c) implies

$$\varphi_n \to \varphi \quad \text{uniformly on} \quad [0,b_1]$$

and

$$\varphi'(x) = \lim_{n\to\infty} \varphi_n'(x) = \psi(x) \ . \tag{2.22.d}$$

In particular, taking $x = a$, (2.22.d) gives (2.21).

We continue with the preliminary work needed for the proof of Theorem 2.1.

<u>Lemma 2.23</u>

Assume (r.1), (F.1)-(F.5) are satisfied.

Let $\lambda > 0$ and $0 < a < b < +\infty$ be fixed. If $c \in (a,b)$ then we have

$$\Lambda[a,c] + \Lambda[c,b] < \Lambda[a,b] . \tag{2.24}$$

Moreover, if $\lambda > \mu_1(a,c,0)$ or $\lambda > \mu_1(c,b,0)$ inequality (2.27) is strict.

<u>Proof</u>

Since $\theta = 0$ we suppress θ from the notations, i.e. $\mu_1(\alpha,\beta) = \mu_1(\alpha,\beta,0)$ and $V_+(\lambda,\alpha,\beta,x) = V_+(\lambda,\alpha,\beta,0,x)$. Also, for convenience we adapt the notation that $V_+(\lambda,\alpha,\beta,x) \equiv 0$ when $\lambda < \mu_1(\alpha,\beta)$.

If $\lambda < \mu_1(a,b)$, by (2.3.b), $\Lambda[a,c] = \Lambda[c,b] = \Lambda[a,b] = 0$. Hence (2.24) trivially hold.

If only one of the inequalities $\lambda < \mu_1(a,c)$, $\lambda < \mu_1(c,b)$ is satisfied, say $\lambda < \mu_1(a,c)$, then by (2.3.b), $\Lambda[a,c] = 0$. Hence it follows from (2.5.a) if $b < +\infty$ and from (2.21) if $b = +\infty$ that

$$\begin{aligned}
\Lambda[c,b] &= \int_c^b (V_+'(\lambda,x,b,x))^2 dx \\
&< \int_a^c (V_+'(\lambda,x,b,x))^2 dx + \int_c^b (V_+'(\lambda,x,b,x))^2 dx \\
&= \int_a^b (V_+'(\lambda,x,b,x))^2 dx \\
&= \Lambda[a,b] .
\end{aligned}$$

Likewise $\Lambda[a,c] < \Lambda[a,b]$ if $\lambda < \mu_1(c,b)$.

In the remaining case, both $\Lambda[a,c]$ and $\Lambda[c,b]$ are nonzero. Again by (2.5.a) when $b < +\infty$ and by (2.21) when $b = +\infty$, we obtain

$$\Lambda[a,b] - \Lambda[c,b] = \int_a^c (V_+'(\lambda,x,b,x))^2 dx \tag{2.25}$$

and

$$\Lambda[a,c] = \int_a^c (V_+'(\lambda,x,c,x))^2 dx \,.$$

From (2.3.b), we know $V_+'(\lambda,x,c,x) = 0$ if $\lambda < \mu_1(x,c)$. Hence there is an $\varepsilon > 0$ such that $V_+'(\lambda,x,c,x) = 0$ whenever $x \in [c - \varepsilon, c]$ and consequently we have

$$\Lambda[a,c] = \int_a^{c-\varepsilon} (V_+'(\lambda,x,c,x))^2 dx \,. \tag{2.26}$$

From (1.47), we know, for $x \in [a, c - \varepsilon)$

$$(V_+'(\lambda,x,b,x))^2 > (V_+'(\lambda,x,c,x))^2 \,. \tag{2.27}$$

Combining (2.25), (2.26) and (2.27), we conclude that

$$\Lambda[a,c] + \Lambda[c,b] < \Lambda[a,b] \,.$$

Corollary 2.28

Assume (r.1), (F.1)-(F.5) are satisfied. Then

$$\lim_{a\to\infty} \Lambda[a,\infty] = 0 \,. \tag{2.29}$$

Proof

Let $\alpha > 0$. By Lemma 2.11 and Lemma 2.23, we have

$$0 < \Lambda[a,\infty] < \Lambda[\alpha,\infty] - \Lambda[\alpha,a]$$

for all $\alpha < a < +\infty$. Letting $a \to +\infty$ and invoking Lemma 2.13, we obtain (2.29).

Remark 2.30

Without assuming (F.5) the same kind of technique used in the proof of Corollary 2.7 shows that

$$\lim_{a \to \infty} \Lambda^{\pm}[a,\infty] = 0 .$$

We are first going to prove Theorem 2.1 for the special case of $\theta = 0$. For fixed $\lambda > 0$ and $a > 0$, we denote the function $G_1(x)$ by

$$G_1(x) = \Lambda[a,x] + \Lambda[x,\infty] \qquad (2.31)$$

for $x \in [a,\infty)$ and define $G_1(+\infty) = \lim_{x \to +\infty} G_1(x)$.

The function G_1 has the following properties.

Lemma 2.32

Suppose (r.1), (F.1)-(F.5) are satisfied. Then G_1 is continuously differentiable on $[a,\infty)$ such that

$$G_1(+\infty) = \Lambda[a,\infty] \qquad (2.33)$$

and

$$G_1'(x) = [V_+'(\lambda,a,x,0,x)]^2 - [V_+'(\lambda,x,\infty,0,x)]^2 . \qquad (2.34)$$

Proof

$G_1 \in C^1[a,\infty)$ is an immediate consequence of Propositions 2.4, 2.20 and (2.31). Next, by Lemma 2.13 and Corollary 2.28, we have (2.33). Finally (2.34) follows from (2.5.b) and (2.21).

We are now able to find a one-node solution for $\theta = 0$ in (1.1.b). From now on until the end of Remark 2.51, we suppress the θ dependence in our notation.

Theorem 2.35

Suppose (r.1), (F.1)-(F.5) are satisfied. Let $\lambda > 0$, $a > 0$ be given and $\theta = 0$ in (1.1.b) be fixed. Then $S^+_{a,2}(\lambda)$ and $S^-_{a,2}(\lambda)$ are nonempty.

Proof

By (1.41.c), there is an $\varepsilon > 0$ such that $\lambda < \mu_1(a,x)$ for $x \in [a,a+\varepsilon]$. Hence, by Lemma 2.11 (i) and (2.31), $G_1(a) = \Lambda[a,\infty]$. From Proposition 1.43 (i) and (2.34) we know $G_1'(x) = -[V_+'(\lambda,x,\infty,x)]^2 < 0$ for $x \in [a,a+\varepsilon]$. Furthermore, by (2.33), $G_1(+\infty) = \Lambda[a,\infty]$. We conclude that G_1 must attain its infimum at some point $z \in (a+\varepsilon,\infty)$.

Next, we define

$$u(x) = \begin{cases} V_+(\lambda,a,z,x), & x \in [a,z] \\ V_-(\lambda,z,\infty,x), & x \in [z,\infty) . \end{cases} \tag{2.36}$$

We claim $u \in C^1[a,\infty)$. Obviously,

$$u(z) = V_+(\lambda,a,z,z) = V_-(\lambda,z,\infty,z) = 0 .$$

Since $G'(z) = 0$, by (2.34) and $V_+ = -V_-$, we have

$$\lim_{x \to z^-} u'(x) = V_+'(\lambda,a,z,z) = V_-'(\lambda,z,\infty,z) = \lim_{x \to z^+} u'(x) .$$

Finally, it can be easily checked that $u(x)$ satisfies $(I)_a$. Thus $u \in S^+_{a,2}(\lambda)$ and hence $-u \in S^-_{a,2}(\lambda)$.

Remark 2.37

The proof shows that, for given $a > 0$, G_1 attains its minimum at an interior point of (a,∞).

To obtain n-node solutions, more work is needed. Let $a > 0$, then for $n \in \mathbb{N}$ we denote the set A_n by

$$A_n = \{(x_1,x_2,\ldots,x_n) | a \le x_1 \le x_2 \le \cdots \le x_n < +\infty\} .$$

Next, put $x_0 = a$ and $x_{n+1} = +\infty$. Then, for fixed $\lambda > 0$, we define the function G_n on A_n by

$$G_n(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n+1} \Lambda[x_{i-1},x_i] . \tag{2.38}$$

<u>Lemma 2.39</u>

Assume (r.1), (F.1)-(F.5) are satisfied. Then G_n is continuously differentiable on A_n and

$$\frac{\partial G_n}{\partial x_i}(x_1,x_2,\ldots,x_n) = [V'_+(\lambda,x_{i-1},x_i,0,x_i)]^2$$

$$- [V'_+(\lambda,x_i,x_{i+1},0,x_i)]^2 . \tag{2.40}$$

<u>Proof</u>

It immediately follows from Propositions 2.4, 2.20 and (2.38).

<u>Lemma 2.41</u>

Assume (r.1), (F.1)-(F.5) are satisfied. Let $n > 1$, then

(i) If $x_1 = a$, $G_n(x_1,x_2,\ldots,x_n) = G_{n-1}(x_2,\ldots,x_n)$.

(ii) If $x_i = x_{i+1}$, $G_n(x_1,x_2,\ldots,x_n) = G_{n-1}(x_2,\ldots,x_i,x_{i+2},\ldots,x_n)$.

(iii) If $\{(x_1(m),x_2(m),\ldots,x_n(m))\} \subset A_n$ is a sequence such that

$$\lim_{m\to\infty} x_i(m) = z_i < +\infty \quad \text{for} \quad 0 \le i < k$$

and

$$\lim_{m\to\infty} x_i(m) = +\infty \quad \text{for} \quad k \le i \le n .$$

Then

$$\lim_{m\to\infty} G_n(x_1(m),x_2(m),\ldots,x_n(m)) = \begin{cases} G_{k-1}(z_1,z_2,\ldots,z_{k-1}) & \text{if} \quad k > 1 \\ \Lambda[a,\infty] & \text{if} \quad k = 1 . \end{cases}$$

Proof

By (1.41.c) and Lemma 2.11 (i), we have $\Lambda[x,x] = 0$ for $x \in [a,\infty)$. This together with (2.38) implies (i) and (ii). To prove (iii) we know that Proposition 2.4 implies

$$\lim_{m\to\infty} \sum_{i=1}^{k-1} \Lambda[x_{i-1}(m),x_i(m)] = \sum_{i=1}^{k-1} \Lambda[z_{i-1},z_i]$$

where $z_0 = x_0(m) = a$. Next, by Proposition 2.4 and Lemma 2.13

$$\lim_{m\to\infty} \Lambda[x_{k-1}(m),x_k(m)] = \Lambda[x_{k-1},\infty]$$

Thus, letting $x_{n+1}(m) = +\infty$, it remains to prove

$$\lim_{m\to\infty} \sum_{i=k+1}^{n+1} \Lambda[x_{i-1}(m),x_i(m)] = 0 \ . \tag{2.42}$$

From Lemmas 2.11 and 2.23, it is easy to see that for all $m \in \mathbb{N}$

$$0 < \sum_{i=k+1}^{n+1} \Lambda[x_{i-1}(m),x_i(m)] < \Lambda[x_{i-1}(m),\infty] \ . \tag{2.43}$$

Letting $m \to +\infty$, (2.42) follows from (2.43) and (2.29).

We are now ready to establish n-node solutions for $n > 2$ in the case $\theta = 0$.

Theorem 2.44

Suppose (r.1), (F.1)-(F.5) are satisfied. Let $\lambda > 0$ and $a > 0$ be given, then $S^+_{a,n}(\lambda,0)$ and $S^-_{a,n}(\lambda,0)$ are nonempty for all $n \in \mathbb{N}$.

Proof

For convenience, we suppress the θ dependence from the notation $V_\pm$. Since $\Lambda[x_{i-1},x_i] > 0$ and at least one of them is positive, we know

$$G_k(x_1,x_2,\ldots,x_k) > 0 \quad \text{for all} \quad (x_1,x_2,\ldots,x_k) \in A_k$$

and hence $\inf\limits_{A_k} G_k(x_1,x_2,\ldots,x_k)$ exists.

By an interior point of A_k, we mean a point $(x_1,x_2,\ldots,x_k) \in A_k$ such that $a < x_1 < x_2 < \cdots < x_k < +\infty$. We will show G_k attains its global infimum at an interior point $(z_1,z_2,\ldots,z_k)$ of A_k. Assuming that for now and letting $z_0 = a$ and $z_{k+1} = +\infty$, from (2.40), we have, for $i = 1,2,\ldots,k$, that

$$\frac{\partial G_k}{\partial x_i}(z_1,z_2,\ldots,z_k) = [V_+'(\lambda,z_{i-1},z_i,z_i)]^2 - [V_+'(\lambda,z_i,z_{i+1},z_i)]^2 = 0$$

that is

$$|V_+'(\lambda,z_{i-1},z_i,z_i)| = |V_+'(\lambda,z_i,z_{i+1},z_i)|, \; i = 1,2,\ldots,k \; . \qquad (2.45)$$

Also, note that if $\lambda \leq \mu_1(z_{j-1},z_j)$ for some j, let $\ell \geq j$ be the largest value such that $\lambda \leq \mu_1(z_{\ell-1},z_\ell)$ and $\lambda > \mu_1(z_\ell,z_{\ell+1})$. But this implies $V_+'(\lambda,z_{\ell-1},z_\ell,z_\ell) = 0$ and $V_+'(\lambda,z_\ell,z_{\ell+1},z_\ell) \neq 0$ which contradicts (2.45). Thus $\lambda > \mu_1(z_{i-1},z_i)$ for $i = 1,2,\ldots,k+1$, and if we put

$$u(x) = (-1)^i V_+(\lambda,z_i,z_{i+1},x) \quad \text{for} \quad x \in [z_i,z_{i+1}) \; ,$$

$i = 0,1,2,\ldots,k$, then $u \in C^1[a,\infty)$ is the desired k-node solution with nodes $z_1,z_2,\ldots,z_k$.

Thus it remains to show that G_k attains its infimum at an interior point $(z_1,z_2,\ldots,z_k)$ of A_k. To achieve this goal it is sufficient to prove, by induction, the following statement:

If, for $1 \leq k \leq n-1$,

G_k attains its global minimum at an interior point of A_k (2.46.a)

and

$$\operatorname*{Min}_{A_k} G_k(x_1,x_2,\ldots,x_k) > \operatorname*{Inf}_{A_{k+1}} G_{k+1}(x_1,x_2,\ldots,x_{k+1}) \tag{2.46.b}$$

then these statements also hold for $k = n$.

It is clear, from Remark 2.37, that (2.46.a) holds for $k = 1$. Also, if G_1 attains its global minimum at $z \in (a,\infty)$, then from Lemma 2.23 there exists a $t \in (z,\infty)$ such that

$$\Lambda[z,t] + \Lambda[t,\infty] < \Lambda[z,\infty]$$

hence

$$\begin{aligned}\operatorname*{Min}_{A_1} G_1(x) &= G_1(z) = \Lambda[a,z] + \Lambda[z,\infty] \\ &> \Lambda[a,z] + \Lambda[z,t] + \Lambda[t,\infty] \\ &= G_2(z,t) > \operatorname*{Inf}_{A_2} G_2(x_1,x_2)\end{aligned}$$

which gives (2.46.b) for $k = 1$.

Next, if (2.46.a) and (2.46.b) hold for $1 < k < n - 1$ and suppose (2.46.a) is false for $k = n$, then there exists a sequence $\{(S_1(m),S_2(m),\ldots,S_n(m))\} \subset A_n$ such that

$$\lim_{m\to\infty} S_i(m) = t_i \quad \text{(might be } +\infty) \qquad 1 < i < n \tag{2.47}$$

and

$$\operatorname*{Inf}_{A_n} G_n(x_1,x_2,\ldots,x_n) = \lim_{m\to\infty} G_n(S_1(m),S_2(m),\ldots,S_n(m)) \,. \tag{2.48}$$

Suppose $t_n < +\infty$. Then, letting $t_0 = a$, there exists $0 < j < n - 1$ such that

$$t_j = t_{j+1} \,.$$

Hence, by Lemma 2.41 (i) or (ii), we have

$$G_n(t_1,t_2,\ldots,t_n) = G_{n-1}(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n) \,. \tag{2.49}$$

From Lemma 2.39, (2.47) and $t_n < +\infty$, we know

$$\lim_{m\to\infty} G_n(S_1(m),S_2(m),\ldots,S_n(m)) = G_n(t_1,t_2,\ldots,t_n) \; . \qquad (2.50)$$

Applying the induction hypothesis and combining (2.48)-(2.50), this leads to

$$\operatorname*{Inf}_{A_n} G_n < \operatorname*{Min}_{A_{n-1}} G_{n-1} = \operatorname*{Inf}_{A_{n-1}} G_{n-1} \leq G_{n-1}(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)$$

$$= G_n(t_1,t_2,\ldots,t_n) = \lim_{m\to\infty} G_n(S_1(m),\ldots,S_n(m)) = \operatorname*{Inf}_{A_n} G_n$$

which is obviously absurd. If $t_n = +\infty$. Let $0 \leq j \leq n-1$ be the largest value such that $t_j < +\infty$. Suppose $j > 0$. By Lemma 2.41 (iii) we have

$$\lim_{m\to\infty} G_m(S_1(m),\ldots,S_n(m)) = G_j(t_1,\ldots,t_j) \; .$$

Applying the induction hypothesis, we obtain the same sort of contradiction. Thus it remains the case $j = 0$. In this case, by Lemma 2.41 (iii)

$$\lim_{m\to\infty} G_n(S_1(m),\ldots,S_n(m)) = \Lambda[a,\infty] \; .$$

From Lemma 2.23, there exists a $z \in (a,\infty)$ such that

$$\Lambda[a,\infty] > \Lambda[a,z] + \Lambda[z,\infty] = G_1(z) \geq \operatorname*{Inf}_{A_1} G_1 \; .$$

Thus,

$$\operatorname*{Inf}_{A_n} G_n = \lim_{m\to\infty} G_n(S_1(m),\ldots,S_n(m)) = \Lambda[a,\infty] > \operatorname*{Inf}_{A_1} G_1 = \operatorname*{Min}_{A_1} G_1$$

which is contrary to the induction hypothesis again.

Finally, arguing like the case $k = 1$, we can prove (2.46.b) holds for $k = n$ and this completes the proof.

Remark 2.51

(a) Actually any critical point of G_n gives a n-node solution. However, if the n-node solution is unique (up to the sign) then $(z_1, z_2, \ldots, z_n)$ must be the point at which the minimum of G_n occurs where $z_1, z_2, \ldots, z_n$ are nodes of the n-node solution.

(b) For fixed $\lambda > 0$ and $a > 0$, let $\Gamma_1 = \Lambda[a,\infty]$ and for $n > 2$, $\Gamma_n = \inf_{A_{n-1}} G_{n-1}$. Then by (2.46)

(i) $\Gamma_1 > \Gamma_2 > \cdots > \Gamma_n > \cdots > 0$.

(ii) Let u be a solution of $(I)_a$ and

$$J(u) = \int_a^\infty \lambda r(x) u^2(x) - (u'(x))^2 - 2 \int_0^{u(x)} f(x,y)dydx .$$

If $J(u) < \Gamma_n$ then u has at least n interior zeroes in (a,∞).

Next, with the aid of Proposition 1.76, we know

(iii) Γ_n are continuous functions of λ and a.

Moreover, it is not difficult to prove, from Proposition 2.20, Corollaries 1.17 and 1.45, that

(iv) For fixed $a > 0$, Γ_n are increasing functions of λ and for fixed $\lambda > 0$, Γ_n are decreasing functions of a.

To complete the proof of Theorem 2.1 for the case $0 < \theta < \frac{\pi}{2}$ we need the following technical lemma.

Lemma 2.52

Assume (r.1) and (F.1)-(F.4) are satisfied. Let $0 < x_0 < x_1 < \cdots < x_n < x_{n+1}$. Suppose $x_{n+1} < +\infty$ (resp. $= +\infty$). Let τ be a function defined by

$$\tau(n) = \begin{cases} + & \text{if } n \text{ is odd} \\ - & \text{if } n \text{ is even} \end{cases} \tag{2.53}$$

and $V_{\tau(n)}(\lambda,a,b,\theta,\cdot)$ be the positive or negative (depending on $\tau(n)$) solution as mentioned in Remark 1.6 (b). For fixed θ_1, $0 < \theta_1 < \frac{\pi}{2}$ suppose $\lambda > \mu_1(x_0,x_1,\theta_1)$ and $\lambda > \underset{1<i<n}{\text{Max}}\ \mu_1(x_i,x_{i+1},0)$ (resp. $\underset{1<i<n-1}{\text{Max}}\ \mu_1(x_i,x_{i+1},0)$) where $\mu_1(a,b,\theta)$ was defined as in (1.39) and (1.40). Suppose for $2 < i < n$

$$|V'_{\tau(i)}(\lambda,x_{i-1},x_i,0,x_i)| > |V'_{\tau(i+1)}(\lambda,x_i,x_{i+1},0,x_i)| \tag{2.54}$$

and

$$|V'_{\tau(1)}(\lambda,x_0,x_1,\theta_1,x_1)| > |V'_{\tau(2)}(\lambda,x_1,x_2,0,x_1)|\ . \tag{2.55}$$

Then there exist $t_0 < t_1 < t_2 < \cdots < t_{n+1}$, $t_0 = x_0$, $t_{n+1} = x_{n+1}$ such that

(i) $t_i < x_i$, $\quad 1 < i < n$ (2.56)

(ii) $|V'_{\tau(1)}(\lambda,t_0,t_1,\theta_1,t_1)| = |V'_{\tau(2)}(\lambda,t_1,t_2,0,t_1)|$ (2.57)

and for $2 < i < n$

$$|V'_{\tau(i)}(\lambda,t_{i-1},t_i,0,t_i)| = |V'_{\tau(i+1)}(\lambda,t_i,t_{i+1},0,t_i)| \tag{2.58}$$

(iii) $|V'_{\tau(1)}(\lambda,t_0,t_1,\theta_1,t_0)| < |V'_{\tau(1)}(\lambda,x_0,x_1,\theta_1,x_0)|$

if $\theta_1 = 0$ (2.59)

(iv) $|V'_{\tau(n+1)}(\lambda,t_n,t_{n+1},0,t_{n+1})| > |V'_{\tau(n+1)}(\lambda,x_n,x_{n+1},0,x_{n+1})|$

if $x_{n+1} < +\infty$ (2.60)

(v) $\lambda > \mu_1(t_0,t_1,\theta_1)$ and $\lambda > \underset{1<i<n}{\text{Max}}\ \mu_1(t_i,t_{i+1},0)$ (resp.

$\underset{1<i<n-1}{\text{Max}}\ \mu_1(t_i,t_{i+1},0)$). (2.61)

Moreover, if $\tau(j)$ is replaced by $\tau(j+1)$ wherever $\tau(j)$ appeared in the above statement the results still hold.

Remark 2.62

In view of the definition of τ, if we define

$$v(x) = \begin{cases} V_{\tau(1)}(\lambda,t_0,t_1,\theta,x) & \text{for } x \in [t_0,t_1) \\ V_{\tau(i+1)}(\lambda,t_i,t_{i+1},0,x) & \text{for } x \in [t_i,t_{i+1}) \quad 1 < i < n \end{cases} \tag{2.63}$$

and let $a = t_0$, $b = t_{n+1}$ then $v \in S^+_{a,b,n+1}(\lambda,\theta)$ (resp. $S^+_{a,n+1}(\lambda,\theta)$) in the $\tau(j)$ case and $v \in S^-_{a,b,n+1}(\lambda,\theta)$ (resp. $S^-_{a,n+1}(\lambda,\theta)$) in the $\tau(j+1)$ case.

Proof

Since θ_1 is considered fixed, we will suppress θ_1 as well as 0 from our notation when there is no confusion.

The proof will proceed by induction. We first look at the case $n = 1$. For given x_0 and x_2, we define functions p^+ and p^- with domain (x_0,x_2) by

$$p^{\pm}(x) = [V'_{\pm}(\lambda,x_0,x,\theta_1,x)]^2 - [V'_{\mp}(\lambda,x,x_2,0,x)]^2 \tag{2.64}$$

where, for convenience, we adapt the notation $V_{\pm}(\lambda,a,b,\theta,x) \equiv 0$ whenever $\lambda < \mu_1(a,b,\theta)$. Then it is clear, from Proposition 1.76, that p^+ and p^- are continuous on (x_0,x_2). From the hypothesis (2.55), we know $p^+(x_1) > 0$. From (1.41.c) and Proposition 1.43 (i), we know, for x near x_0, that

$$p^+(x) = -[V'_-(\lambda,x,x_2,x)]^2 < 0 .$$

Therefore, there is a $t_1 \in (x_0,x_1]$ such that

$$p^+(t_1) = 0 . \tag{2.65}$$

Letting $t_0 = x_0$ and $t_2 = x_2$ this yields (2.56). Combining (2.64) with (2.65), we obtain (2.57).

Supppose $x_2 < +\infty$. Now $t_1 < x_1$ and since $\mu_1(a,b,\theta)$ is a decreasing function of b for a and θ fixed, we have

$\lambda > \mu_1(t_1,t_2)$ and hence

$$|V_-^{\prime}(\lambda,t_1,t_2,t_1)| > 0 \ .$$

Combining this inequality with (2.57), we have

$$|V_+^{\prime}(\lambda,t_0,t_1,t_1)| > 0 \ . \tag{2.66}$$

If $\lambda < \mu_1(t_0,t_1)$ the only solution for $(I)_{t_0,t_1}$ would be the trivial solution which is contrary to (2.66). Thus $\lambda > \mu_1(t_0,t_1)$ giving (2.61) if $x_2 < +\infty$.

If $x_2 = +\infty$, $|V_-^{\prime}(\lambda,t_1,\infty,t_1)| > 0$. Consequently the same argument as above shows $\lambda > \mu_1(t_0,t_1)$. Thus we have (2.61) even if $x_2 = +\infty$.

Equation (2.58) is void for $n = 1$. Inequalities (2.59) and (2.60) easily follow from (2.56), (1.47) and (1.49). Thus, with an analogous treatment of $p^-(x)$ which corresponds to the $\tau(j + 1)$ case we complete the case of $n = 1$.

Suppose the result holds for $n = k - 1$. We are going to prove that it is true for $n = k$. Granted that (2.56)-(2.58) are true, we can verify (2.59)-(2.61) by the same reasoning as in the case $n = 1$. Thus, it remains to show (2.56)-(2.58).

First, by applying the induction hypothesis to $k + 1$ ordered points $(x_0,x_1,\ldots,x_k)$, we get $k + 1$ ordered points $(s_0,s_1,\ldots,s_k)$ with the corresponding (2.56)-(2.60) as follows

$$s_0 = x_0, \ s_k = x_k \ \text{and} \ s_i < x_i \ \text{for} \ 1 \le i \le k-1,$$

$$|V_{\tau(i)}^{\prime}(\lambda,s_{i-1},s_i,s_i)| = |V_{\tau(i+1)}^{\prime}(\lambda,s_i,s_{i+1},s_i)| \quad 1 \le i \le k-1 \tag{2.67}$$

$$|V_{\tau(1)}^{\prime}(\lambda,s_0,s_1,s_0)| < |V_{\tau(1)}^{\prime}(\lambda,x_0,x_1,x_0)| \ ,$$

$$|V_{\tau(k)}^{\prime}(\lambda,s_{k-1},s_k,s_k)| > |V_{\tau(k)}^{\prime}(\lambda,x_{k-1},x_k,x_k)| \ . \tag{2.68}$$

From the hypothesis (2.54) and (2.68), we have

$$|V_{\tau(k)}^{\prime}(\lambda,s_{k-1},s_k,s_k)| \ge |V_{\tau(k+1)}^{\prime}(\lambda,x_k,x_{k+1},x_k)| \ . \tag{2.69}$$

Put $s_{k+1} = x_{k+1}$. If equality occurs in (2.69), we have the $n = k$ case. If (2.69) is not an equality, from (2.67) and (2.69), we can apply the induction hypothesis to $(s_1, s_2, \ldots, s_{k+1})$ and obtain $(s_1', \ldots, s_{k+1}')$ with the corresponding (2.56)-(2.60) as follows

$$s_1' = s_1,\ s_{k+1}' = s_{k+1} \quad \text{and} \quad s_i' < s_i \quad \text{for} \quad 2 < i < k \tag{2.70}$$

$$|V_{\tau(i)}'(\lambda, s_{i-1}', s_i', s_i')| = |V_{\tau(i+1)}'(\lambda, s_i', s_{i+1}', s_i')| \quad 2 < i < k \tag{2.71}$$

$$|V_{\tau(2)}'(\lambda, s_1', s_2', s_1')| < |V_{\tau(2)}'(\lambda, s_1, s_2, s_1)| \tag{2.72}$$

$$|V_{\tau(k+1)}'(\lambda, s_k', s_{k+1}', s_{k+1}')| > |V_{\tau(k+1)}'(\lambda, s_k, s_{k+1}, s_{k+1})|$$
$$\text{if} \quad x_{k+1} < +\infty\ . \tag{2.73}$$

Thus, it follows from the case $i = 1$ of (2.67) and (2.72), that

$$|V_{\tau(2)}'(\lambda, s_1', s_2', s_1')| < |V_{\tau(1)}'(\lambda, s_0, s_1, s_1)|\ . \tag{2.74}$$

Put $s_0' = s_0$. Again if equality occurs in (2.74) we are finished; otherwise we repeat the same process on $(s_0', s_1', \ldots, s_k')$. Continuing in this fashion, we define ordered $k + 2$ tuples of points, (x_i), (s_i), (s_i') etc. Either this process terminates in finitely many steps and hence we complete the proof or we have a sequence of $k + 2$ ordered points $\{(T^m(0), T^m(1), \ldots, T^m(k+1))\}$, such that

$$T^m(0) = T^{m+1}(0),\ T^m(k+1) = T^{m+1}(k+1) \quad \text{and} \quad T^m(i+1) < T^m(i)$$
$$\text{for} \quad 1 < i < k \quad \text{and for all} \quad m\ , \tag{2.75}$$

$$|V_{\tau(1)}'(\lambda, T^{m+1}(0), T^{m+1}(1), T^{m+1}(0))| < |V_{\tau(1)}'(\lambda, T^m(0), T^m(1), T^m(0))|\ ,$$
$$|V_{\tau(k+1)}'(\lambda, T^{m+1}(k), T^{m+1}(k+1), T^{m+1}(k+1))|$$
$$> |V_{\tau(k+1)}'(\lambda, T^m(k), T^m(k+1), T^{m+1}(k+1))| \quad \text{if} \quad x_{k+1} < +\infty$$

and if m is odd

$$|V_{\tau(i)}'(\lambda, T^m(i-1), T^m(i), T^m(i))| = |V_{\tau(i+1)}'(\lambda, T^m(i), T^m(i+1), T^m(i))|$$
$$\text{for} \quad 1 < i < k-1 \tag{2.76}$$

and if m is even

$$|V'_{\tau(i)}(\lambda, T^m(i-1), T^m(i), T^m(i))| = |V'_{\tau(i+1)}(\lambda, T^m(i), T^m(i+1), T^m(i))|$$

$$\text{for} \quad 2 < i < k \ . \qquad (2.77)$$

Since (2.75) tells us that for fixed $0 < i < k+1$, $\{T^m(i)\}$ are monotone nonincreasing sequences and bounded below by x_0. Thus

$$\lim_{m\to\infty} T^m(i) = t_i \quad \text{exists for} \quad 0 < i < k+1$$

passing to the limit in (2.75)-(2.77), we get (2.56)-(2.58) for the case $n = k$. The same argument takes care of the $\tau(j+1)$ case. We omit it.

<u>Completion of the proof of Theorem 2.1</u>

Since positive and negative solutions have already been constructed in Theorem 1.2, we only need to consider $n > 2$. Also, the proof of the case of $S^+_{a,n}(\lambda,\theta)$ is the same as that of $S^-_{a,n}(\lambda,\theta)$, so only the first one will be carried out.

By Theorem 2.44, we can pick a $u \in S^+_{a,n}(\lambda,0)$. Let $x_1, x_2, \ldots, x_{n-1}$ be the nodes of u and put $x_0 = a$, $x_{k+1} = +\infty$, then it is clear that

$$u(x) = V_{\tau(i)}(\lambda, x_{i-1}, x_i, 0, x) \quad \text{for} \quad x \in [x_{i-1}, x_i], \quad 1 < i < n \ .$$

By (1.37), we have, for $0 < \theta_1 < \frac{\pi}{2}$, that

$$|V'_+(\lambda, x_0, x_1, \theta_1, x_1)| > |V'_+(\lambda, x_0, x_1, 0, x_1)| \ . \qquad (2.79)$$

Since

$$|V'_+(\lambda, x_0, x_1, 0, x_1)| = |u(x_1)| = |V'_-(\lambda, x_1, x_2, 0, x_1)| \ .$$

Combining this with (2.79) we get

$$|V'_+(\lambda, x_0, x_1, \theta_1, x_1)| > |V'_-(\lambda, x_1, x_2, 0, x_1)| \ . \qquad (2.80)$$

Now, looking at $V_{\tau(1)}(\lambda,x_0,x_1,\theta_1,\cdot)$, $V_{\tau(i)}(\lambda,x_{i-1},x_i,0,\cdot)$, $2 < i < n$ and with the inequality (2.80), we satisfy the hypothesis of Lemma 2.52 from which the required (n - 1)-node solution follows.

§3. EXISTENCE OF SOLUTIONS WITH A PRESCRIBED NUMBER OF NODES WHEN THE NONLINEARITY IS ODD ONLY NEAR ZERO

The goal of this section is to give an existence result for nodal solutions of problem $(I)_a$ under weaker assumptions than earlier. From Theorems 1.2 and 1.5, we already know that, for every $\lambda > 0$, there exist a unique positive and a unique negative solution. However, we have left open the question of whether or not there exist solutions with nodes without assuming (F.5). We will give an affirmative answer here, provided $F(x,y)$ is symmetric in a neighborhood of $(x,y) = (+\infty,0)$ in the xy-plane, that is,

(F.5)' There are positive numbers δ and X such that

$$F(x,-y) = F(x,y) \quad \text{for} \quad x \in [X,\infty) \quad \text{and} \quad |y| < \delta.$$

<u>Theorem 3.1</u>

Assume (r.1), (F.1)-(F.4) and (F.5)' are satisfied. Let $\lambda > 0$, $a > 0$ and $0 < \theta < \frac{\pi}{2}$ be given, then $S^+_{a,n}(\lambda,\theta)$ and $S^-_{a,n}(\lambda,\theta)$ are nonempty for all $n \in \mathbb{N}$.

<u>Proof</u>

Suppose u is a solution of $(I)_\alpha$. By Corollary 1.72

$$\|u\|_{L^\infty[\alpha,\infty)} < K_3(\lambda,\alpha) \;.$$

From (1.75.b), we can find an $\alpha_1 > X$ such that $K_3(\lambda,\alpha_1) < \delta$. Hence u is a solution of $(I)_{\alpha_1}$ if and only if it is a solution of the problem

$$-u'' = (\lambda r(x) - H(x,u))u, \quad \alpha_1 < x < +\infty, \tag{3.2.a}$$

$$u(\alpha_1)\cos\theta - u'(\alpha_1)\sin\theta = 0, \quad u \in L^2[\alpha_1,\infty) \tag{3.2.b}$$

where the function H is defined by

$$H(x,y) = \begin{cases} F(x,y) & \text{if } y > 0, \quad x > 0 \\ F(x,-y) & \text{if } y < 0, \quad x > 0 . \end{cases}$$

By Theorem 2.1, $S^+_{\alpha_1,n}(\lambda,\theta) \neq \emptyset$ and $S^-_{\alpha_1,n}(\lambda,\theta) \neq \emptyset$ for all n and $0 < \theta < \frac{\pi}{2}$. Pick a $u \in S^+_{\alpha_1,n}(\lambda,0)$ and let $z_1,z_2,\ldots,z_{n-1}$ be its nodes. Then it is clear that

$$u(x) = V_{\tau(i)}(\lambda,z_{i-1},z_i,0,x) \quad \text{for } x \in [z_{i-1},z_i], \quad 1 < i < n$$

where $z_0 = \alpha_1$ and $z_n = +\infty$. By (1.49)

$$|V'_+(\lambda,\alpha_1,z_1,0,z_1)| < |V'_+(\lambda,a,z_1,0,z_1)| . \tag{3.3}$$

From (1.37), we have, for $0 < \theta < \frac{\pi}{2}$, that

$$|V'_+(\lambda,a,z_1,0,z_1)| < |V'_+(\lambda,a,z_1,\theta,z_1)| . \tag{3.4}$$

Combining (3.3) with (3.4) we get

$$|V'_+(\lambda,\alpha_1,z_1,0,z_1)| < |V'_+(\lambda,a,z_1,\theta,z_1)| .$$

Since $V'_+(\lambda,\alpha_1,z_1,0,z_1) = V'_-(\lambda,z_1,z_2,0,z_1)$,

$$|V'_-(\lambda,z_1,z_2,0,z_1)| < |V'_+(\lambda,a,z_1,\theta,z_1)| . \tag{3.5}$$

Let $x_0 = a$ and $x_i = z_i$, $1 < i < n$. Now, looking at $V_{\tau(1)}(\lambda,x_0,x_1,\theta,\cdot)$, $V_{\tau(i)}(\lambda,x_{i-1},x_i,0,\cdot)$, $2 < i < n$, and with the inequality (3.5) we satisfy the hypothesis of Lemma 2.52 from which we obtain the required solution in $S^+_{a,n}(\lambda,\theta)$.

$S^-_{a,n}(\lambda,\theta)$ can be treated similarly. This completes the proof.

§4. UNIQUENESS OF SOLUTIONS WITH A PRESCRIBED NUMBER OF NODES

In this section, we will use a shooting argument to prove uniqueness results for solutions with a prescribed number of nodes, that is, for given $\lambda > 0$, $a > 0$ (and $b > a$ in the bounded domain cases), $S^+_{a,n}(\lambda)$ (resp. $S^+_{a,b,n}$) as well as $S^-_{a,n}(\lambda)$ (resp. $S^+_{a,b,n}(\lambda)$) contains at most one element (see Theorems 4.6, 4.7). Note that the existence of multiple node solutions has already been established in Theorems 2.1 and 3.1. Now uniqueness will be proved for a subclass of such problem. However, the symmetricity assumption (F.5) or (F.5)' will not be assumed. To do so, we consider the following initial value problems

$$-u''(x) = \lambda r(x)u(x) - f(x,u(x)), \quad a < x < +\infty , \tag{4.1.a}$$

$$\left.\begin{array}{ll} u(a) = 0,\ u'(a) = \xi & \text{if } \theta = 0 \text{ in } (1.2) \\ u(a) = \xi,\ u'(a) = \xi \cdot \cot\theta & \text{if } 0 < \theta < \frac{\pi}{2} \text{ in } (1.2) \ . \end{array}\right. \tag{4.1.b}$$

It is assumed that

(f.2) $f(x,y)$ is continuously differentiable in $[0,\infty) \times \mathbf{R}$.

λ is the eigenvalue parameter. Given a, λ, θ and ξ there is a unique solution $U_a(\lambda,\xi,\theta,\cdot)$ of (4.1) which is understood to be extended to its maximal interval of definition. Clearly $U_a(\lambda,\xi,\theta,x)$ is of class C^1 in all of its arguments. Assuming (F.3), we have $U_a(\lambda,0,0,x) \equiv 0$ and for $\xi \neq 0$

$$U_a^2(\lambda,\xi,\theta,\cdot) + U_a'^2(\lambda,\xi,\theta,\cdot) > 0 \ . \tag{4.2}$$

We will assume $r(x) > 0$, $F(x,y) > 0$ and $0 < \theta < \frac{\pi}{2}$. Therefore if $\lambda < 0$, by (4.1.b) $U_a'' \cdot U_a > 0$. From (4.1.b), we know if $\xi \neq 0$,

$U'_a \cdot U_a > 0$ for $x \in (a,a+\varepsilon)$ and some $\varepsilon > 0$. Thus, if $U_a > 0$ for $x \in (a,a+\varepsilon)$ then $U'_a > 0$ and $U_a > 0$ in $(a,a+\varepsilon)$. Hence U'_a and U_a are nondecreasing and never vanish in (a,∞). An analogous argument shows U_a cannot equal zero in (a,∞) either if $U_a < 0$ in $(a,a+\varepsilon)$. So we will be only interested in $\lambda > 0$. Let

$$\mathcal{D}_0^+ \text{ (resp. } \mathcal{D}_0^-) = \{(\lambda,\xi) | \lambda > 0,\ \xi > 0 \text{ (resp. } < 0)\} .$$

For fixed $0 < \theta < \frac{\pi}{2}$, let $\mathcal{D}^+_{a,n}(\theta)$ (resp. $\mathcal{D}^-_{a,n}(\theta)$), $n > 1$, be the set of $(\lambda,\xi) \in \mathcal{D}_0^+$ (resp. $\mathcal{D}_0^-$) such that $U_a(\lambda,\xi,\theta,\cdot)$ has at least n zeroes in (a,∞). Ordering these zeroes as an increasing sequence

$$a < z_{a,1}(\lambda,\xi,\theta) < z_{a,2}(\lambda,\xi,\theta) < \cdots < z_{a,n}(\lambda,\xi,\theta) < \cdots$$

we obtain functions $z_{a,n}$ $(n = 1,2,\ldots)$ such that for fixed $0 < \theta < \frac{\pi}{2}$ and for every integer n. $z_{a,n}$ is defined on $\mathcal{D}^+_{a,n}(\theta)$ (resp. $\mathcal{D}^-_{a,n}(\theta)$). Given $(\lambda,\xi) \in \mathcal{D}^+_{a,n}(\theta)$ (resp. $\mathcal{D}^-_{a,n}(\theta)$), $x = z_{a,n}(\lambda,\xi,\theta)$ solves the equation $U_a(\lambda,\xi,\theta,x) = 0$. From (4.2), $U'_a(\lambda,\xi,\theta,x) \neq 0$ when $x = z_{a,n}(\lambda,\xi,\theta)$. Hence, by implicit function theorem there is a neighborhood $\mathcal{O}_1$ of (λ,ξ) on which $z_{a,n}$ is of class C^1 in its arguments and there exists a maximal open set $\mathcal{O}$ containing $\mathcal{O}_1$ such that $z_{a,n}$ is so on $\mathcal{O}$. Since $\mathcal{D}^+_{a,n}(\theta)$ (resp. $\mathcal{D}^-_{a,n}(\theta)$) is the union of those components $\mathcal{O}$ it is an open set. For fixed $\lambda > 0$, we denote

$$D^+_{a,n}(\lambda,\theta) \text{ (resp. } D^-_{a,n}(\lambda,\theta)) = \{\xi | (\lambda,\xi) \in \mathcal{D}^+_{a,n}(\theta) \text{ (resp. } \mathcal{D}^-_{a,n}(\theta))\} .$$

Then it is easy to see that

$$D^\pm_{a,n}(\lambda,\theta) \supset D^\pm_{a,n+1}(\lambda,\theta) . \tag{4.3}$$

We will use the above notation in the next section. Now, throughout this section unless otherwise stated, we assume $r(x)$

satisfies

(r.2) $r(x) = 1, \quad 0 < x < \infty$

and focus on the special nonlinearity which satisfies

(F.6) There are $\psi_1, \psi_2 \in C^1([0,\infty),[0,\infty))$, $\psi_1(0) = \psi_2(0) = 0$, $\psi_1' > 0$, $\psi_2' > 0$ in $(0,\infty)$, and a positive number σ such that

$$F(x,y) = \begin{cases} \psi_1(w(x)|y|^\sigma), & y > 0, \quad x \in [0,\infty) \\ \psi_2(w(x)|y|^\sigma), & y < 0, \quad x \in [0,\infty) \end{cases}$$

where $w \in C^1([0,\infty),(0,\infty))$.

Furthermore, we list several assumptions related to the functions ψ and w

(ψ.1) There are positive numbers p_1, p_2, q_1 and q_2 such that $\psi_1(t) > p_1 \cdot t^{q_1}$ and $\psi_2(t) > p_2 \cdot t^{q_2}$ for $t \in [0,\infty)$.

(ψ.2) $\psi_1 = \psi_2$.

(ψ.3) $\lim\limits_{t\to\infty} \psi_1(t) = \lim\limits_{t\to\infty} \psi_2(t) = +\infty$.

(w.1) $\dfrac{w'}{w}$ is nondecreasing on $[0,\infty)$.

(w.2) There exists a $\hat{b} \in [0,\infty)$ such that $w'(\hat{b}) > 0$.

Remark 4.4

Hypotheses (w.1) together with (w.2) imply w grows exponentially on $[\hat{b},\infty)$. Hence $\int_0^\infty w^{-2/\sigma} dx < +\infty$ is clearly satisfied and it is easy to check that assumptions (F.6), (ψ.1), (w.1) and (w.2) are stronger than (F.1)-(F.4).

Thus equation (4.1.a) becomes

$$-u'' = \begin{cases} \lambda u - \psi_1(w(x)|u|^\sigma)u & \text{if } u > 0 \\ \lambda u - \psi_2(w(x)|u|^\sigma)u & \text{if } u < 0\,. \end{cases} \tag{4.5}$$

Our first main goal is to prove the following uniqueness results:

Theorem 4.6

Assume (r.2), (F.6), (ψ.3) and (w.1) are satisfied. Let $b > a > 0$ and $n \in \mathbb{N}$. If $\lambda > \mu_n(a,b,0)$, $S^{\pm}_{a,b,n}(\lambda,0)$ has a unique element. If $0 < \theta < \frac{\pi}{2}$, $\lambda > \mu_n(a,b,\theta)$ and $w'(a) > 0$ then $S^{\pm}_{a,b,n}(\lambda,\theta)$ has a unique element.

Theorem 4.7

Assume (r.2), (F.6), (ψ.1), (w.1) and (w.2) are satisfied. Let $a > 0$ and $n \in \mathbb{N}$. Then, for every $\lambda > 0$, $S^{\pm}_{a,n}(\lambda,0)$ contains at most one element. If $w'(a) > 0$, $S^{\pm}_{a,n}(\lambda,\theta)$ contains at most one element for all $0 < \theta < \frac{\pi}{2}$.

Remark 4.8

(a) If case $0 < \theta < \frac{\pi}{2}$, the need for the extra assumption $w'(a) > 0$ will be seen in the proof. It is worth pointing out that Corollary 1.45 (v) has the same sort of assumption and will be used in the proof.

(b) Combining Theorem 4.7 with Theorem 2.1, we have

Corollary 4.9

Assume (r.2), (F.6), (ψ.1), (ψ.2), (w.1) and (w.2) are satisfied. Let $a > 0$ and $n \in \mathbb{N}$. Then for every $\lambda > 0$, $S^{\pm}_{a,n}(\lambda,0)$ has a unique element. If $w'(a) > 0$, $S^{\pm}_{a,n}(\lambda,\theta)$ has a unique element for all $0 < \theta < \frac{\pi}{2}$.

Proof

From Remark 4.4, we know the hypotheses of Theorem 2.1 are satisfied. Thus the result simply follows from Theorems 2.1 and 4.7.

Our second goal in this section is to prove a bifurcation result. Note that for the case of bounded intervals the existence of continua of solutions bifurcating from the eigenvalues of the linearized problem is known for a wide class of nonlinear Strum-Liouville eigenvalue problems (e.g. see [24], [37]). For our setting, Theorem 4.6 shows that these continua are actually differentiable curves in the Banach space $\mathbf{R} \times C^2[a,b]$. Heinz [7] has a result in this spirit. However, our result is applicable to more general nonlinearities and boundary conditions.

Theorem 4.10

Assume the hypotheses of Theorem 4.6 are satisfied. Let $0 < \theta < \frac{\pi}{2}$ be fixed. Then, for each $n \in \mathbf{N}$, $(I)_{a,b}$ possesses two C'-curves of solutions C_n^+ and C_n^- in $\mathbf{R} \times C^2[a,b]$, where $C_n^{\pm} = \{(\lambda, u_n^{\pm}(\lambda)) | \lambda > \mu_n\} \cup \{(\mu_n, 0)\}$ and $u_n^{\pm}(\lambda) \in S^{\pm}_{a,b,n}(\lambda,\theta)$, μ_n being defined in (1.41).

For the unbounded interval case, we also generalize Heinz's result [7].

Theorem 4.11

Assume the hypothesis of Corollary 4.9 are satisfied. Let $0 < \theta < \frac{\pi}{2}$ be fixed. Let E be the Banach space $H^1[a,\infty) \cap L^\infty[a,\infty)$. Then, for each $n \in \mathbf{N}$, $(I)_a$ possesses two curves of solutions C_n^+ and C_n^- in $\mathbf{R} \times E$, with $C_n^{\pm} = \{(\lambda, u_n^{\pm}(\lambda)) | \lambda > 0\} \cup \{(0,0)\}$ and $u_n^{\pm}(\lambda) \in S^{\pm}_{a,n}(\lambda,\theta)$.

We need some preliminary work to prove Theorems 4.6 and 4.7.

Proposition 4.12

Let $\lambda > 0$ and $a > 0$ be fixed. Assume (r.2), (F.6), (w.1), (ψ.3) are satisfied and if $0 < \theta < \frac{\pi}{2}$, then $w'(a) > 0$. Then, for every integer $n > 1$, there exists a positive number $\Omega^+_{a,n}(\lambda,\theta)$ (resp. negative number $\Omega^-_{a,n}(\lambda,\theta)$) such that

$D^+_{a,n}(\lambda,\theta) = (0,\Omega^+_{a,n}(\lambda,\theta))$ (resp. $D^-_{a,n}(\lambda,\theta) = (\Omega^-_{a,n}(\lambda,\theta),0)$).

Moreover, in $(0,\Omega^+_{a,n}(\lambda,\theta)$ (resp. $(\Omega^-_{a,n}(\lambda,\theta),0)$) we have

$$\text{(i)} \quad \frac{\partial z_{a,n}}{\partial \xi} > 0 \quad (\text{resp. } < 0) \tag{4.13}$$

and

$$\text{(ii)} \quad \lim_{\xi \to \Omega^{\pm}_{a,n}(\lambda,\theta)} z_{a,n}(\lambda,\xi,\theta) = +\infty . \tag{4.14}$$

Remark 4.15

(a) When λ, θ, or a is considered fixed, we suppress it from our notation $\Omega^{\pm}_n$ and $D^{\pm}_n$.

(b) If (ψ.2) is satisfied, it is clear that $\Omega^-_n = -\Omega^+_n$. In this case, we use the notation Ω_n instead of Ω^+_n.

(c) It is also clear from Proposition 4.12 and (4.3), that $\Omega^+_{n+1} < \Omega^+_n$ and $\Omega^-_{n+1} > \Omega^-_n$.

(d) Ω^+_n (resp. Ω^-_n) could be $+\infty$ (resp. $-\infty$). For instance, taking $w(x) \equiv c$ (a positive constant) then it can be shown that Ω^+_n (resp. Ω^-_n) $= +\infty$ (resp. $-\infty$) for all $n > 1$.

Since a, λ and θ are considered fixed in Proposition 4.12, they will be suppressed from some of our notation, that is,

$$z_n(\xi) = z_{a,n}(\lambda,\xi,\theta) ,$$

$$U(\xi,\cdot) = U_a(\lambda,\xi,\theta,\cdot) ,$$

$$D_n^{\pm} = D_{a,n}^{\pm}(\lambda,\theta) .$$

We define

$$\varphi(\xi,x) = \frac{\partial}{\partial\xi} U(\xi,x) \tag{4.16}$$

for $x > a$ such that $U(\xi,x)$ is defined. As is well-known [35], $\varphi(\xi,\cdot)$ then is the unique solution of the initial value problem

$$-\varphi'' = \begin{cases} [\lambda - \psi_1(w(x)|U(\xi,x)|^{\sigma}) - \sigma\psi_1'(w(x)|U(\xi,x)|^{\sigma})w(x)|U(\xi,x)|^{\sigma}]\varphi \\ \quad \text{if} \quad U > 0 \\ [\lambda - \psi_2(w(x)|U(\xi,x)|^{\sigma}) - \sigma\psi_2'(w(x)|U(\xi,x)|^{\sigma})w(x)|U(\xi,x)|^{\sigma}]\varphi \\ \quad \text{if} \quad U < 0 , \end{cases} \tag{4.17.a}$$

$$\begin{array}{ll} \varphi(a) = 0, \quad \varphi'(a) = 1 & \text{if} \quad \theta = 0 \\ \varphi(a) = 1, \quad \varphi'(a) = \cot\theta & \text{if} \quad 0 < \theta < \frac{\pi}{2} \end{array} \tag{4.17.b}$$

and

$$\varphi' = \frac{\partial}{\partial\xi} U' . \tag{4.18}$$

Next, we introduce two auxiliary quantities $\Phi(\xi,x)$ and $\Psi(\xi,x)$ by

$$\Phi = \varphi'U - U'\varphi , \tag{4.19.a}$$

$$\Psi = \varphi'U' - U''\varphi . \tag{4.19.b}$$

It is easy to check from (4.5) and (4.17.a) that

$$\Phi' = \begin{cases} \sigma\psi_1'(w|U|^{\sigma})w|U|^{\sigma}U\varphi & \text{if} \quad U > 0 \\ \alpha\psi_2'(w|U|^{\sigma})w|U|^{\sigma}U\varphi & \text{if} \quad U < 0 \end{cases} \tag{4.20.a}$$

and

$$\Psi' = \begin{cases} -\psi_1'(w|U|^{\sigma})w'|U|^{\sigma}U\varphi & \text{if} \quad U > 0 \\ -\psi_2'(w|U|^{\sigma})w'|U|^{\sigma}U\varphi & \text{if} \quad U < 0 . \end{cases} \tag{4.20.b}$$

Moreover, from (4.19), (4.1.b) and (4.17.b), we have

$$\Phi(\xi,a) = 0 \quad \text{for all} \quad 0 < \theta < \frac{\pi}{2} \tag{4.21.a}$$

and

$$\Psi(\xi,a) = \begin{cases} \xi & \text{if } \theta = 0 \\ \xi \cdot \cot^2\theta - U''(\xi,a) & \text{if } 0 < \theta < \frac{\pi}{2} . \end{cases} \tag{4.21.b}$$

Now, we prove several lemmas.

Lemma 4.22

Suppose (r.2), (F.6) and (w.1) are satisfied. Then as a function of x, there is a zero of $U(\xi,\cdot)$ between any two zeroes of $\varphi(\xi,\cdot)$.

Proof

This immediately follows from the Sturm Comparison Theorem by comparing equations (4.5) and (4.17.a).

Lemma 4.23

Assume (r.2), (F.6), (w.1) are satisfied and suppose $w'(x) > 0$ for $x \in [a,\infty)$. Let $0 < \theta < \frac{\pi}{2}$ be fixed. If $\xi \in D_1^+$ (resp. D_1^-) then $U''(\xi,a) < 0$.

Proof

Let $b = z_1(\xi)$, the first zero of $U(\xi,\cdot)$. By Theorem 1.5 $U(\xi,x) = V_+(\lambda,a,b,x)$ for $x \in [a,b]$. Thus, by Remark 1.55 (a), (b), we have the lemma provided that the hypotheses imposed there are satisfied. It is easy to see that all of these assumptions except for (F.2) are satisfied. However, from Proposition 1.43 and Remark 1.44, we know $V_+(\lambda,a,b,\theta,\cdot)$ exists for every $b < +\infty$ such that $\lambda > \mu_1(a,b,\theta)$, $\theta \in [0, \frac{\pi}{2}]$. So we have completed the proof.

Lemma 4.26

Assume (r.2), (F.6) and (w.1) are satisfied. Let $0 < \theta < \frac{\pi}{2}$ be fixed. If $U(\xi,\cdot)$ has its first zero at z_1. Then $\varphi(\xi,x) \neq 0$ for $x \in [a,z_1]$.

Proof

If not, let $\eta > \xi > 0$. Then $\varphi_1 = \eta \cdot \varphi$ is also a solution of equation (4.17.a) and $\varphi_1(a) = \eta$, $\varphi_1'(a) = \eta \cdot \cot\theta$. Put $v = U(\xi,\cdot)$. Since $\varphi_1(a) = \eta > \xi = v(a)$ and φ_1 has a zero in $[a,z_1]$ there exists an x_1 such that $\varphi_1(x_1) = v(x_1)$ and

$$\varphi_1(x) > v(x) > 0 \quad \text{for} \quad x \in [a,x_1) .$$

Thus

$$\varphi_1'(x_1) < v'(x) .$$

Let Φ as defined in (4.19.a). Since from (4.20.a) and the hypothesis $\psi_1' > 0$, $\Phi' > 0$ in $[a,x_1)$, we have

$$\Phi(\xi,x_1) > \Phi(\xi,a) . \tag{4.27}$$

On the other hand, by (4.19.a)

$$\begin{aligned}\Phi(\xi,x_1) &= \varphi'(\xi,x_1)U(\xi,x_1) - U'(\xi,x)\varphi(\xi,x)\\ &= \frac{1}{\eta}\,[\varphi_1'(x_1)v(x_1) - v'(x_1)\varphi_1(x_1)] .\end{aligned}$$

Combining this with $\varphi_1(x_1) = v(x_1) > 0$ and $\varphi_1'(x_1) < v'(x_1)$ we obtain

$$\Phi(\xi,x_1) < 0 .$$

This together with (4.21.a) contradicts (4.27). The case $\xi < 0$ can be treated similarly.

Proposition 4.28

Suppose (r.2), (F.6) and (w.1) are satisfied. Let $\lambda > 0$ and $0 < \theta < \frac{\pi}{2}$ be fixed. Suppose $\xi \in D_n^+$ (resp. D_n^-) for some integer $n \geq 1$, and assume

$$U'(z_k)\varphi(z_k) < 0 \quad (\text{resp. } > 0) \tag{4.29}$$

for $k = 1,2,\ldots,n$ (where $z_k = z_k(\xi)$), then for any $\hat{x} > z_n$ such that $U(x)\varphi(x) \neq 0$ for $z_n < x < \hat{x}$ we have

$$\Psi(\hat{x}) > \Psi(a) + \Phi(\hat{x}) \frac{\Psi(\hat{x}) - \Psi(z_n)}{\Phi(\hat{x}) - \Phi(z_n)}, \qquad (4.30)$$

$$\left(\text{resp.} \quad \Psi(\hat{x}) < \Psi(a) + \Phi(\hat{x}) \frac{\Psi(\hat{x}) - \Psi(z_n)}{\Phi(\hat{x}) - \Phi(z_n)}\right)$$

where $\Psi(\hat{x}) = \Psi(\xi,\hat{x})$ etc.

Proof

We only carry out the case of $\xi \in D_n^+$. Let $z_0 = a$ and $I_k = (z_{k-1}, z_k)$, $k = 1,2,\ldots,n$. Suppose first $\theta = 0$. Since $\varphi(a) = 0$, Lemma 4.22 shows φ cannot vanish in I_1. If $0 < \theta < \frac{\pi}{2}$, Lemma 4.26 implies $\varphi \neq 0$ in I_1 either. Moreover, for all $0 < \theta < \frac{\pi}{2}$, if $k > 2$ the interval I_k contains at most one zero of φ. From (4.29), we know φ changes sign in each I_k for $k > 2$. Also note that $\varphi \neq 0$ in $(z_n,\hat{x})$ by hypothesis. Thus φ has exactly $n - 1$ zeroes $s_1,s_2,\ldots,s_{n-1}$ in $(a,\hat{x})$ and $s_k \in I_{k+1}$ for $1 < k < n - 1$. Define points $x_0,x_1,\ldots,x_{2n}$ by

$$x_0 = a,$$
$$x_{2n} = \hat{x},$$
$$x_{2k-1} = z_k, \qquad 1 < k < n,$$
$$x_{2k} = s_k, \qquad 1 < k < n - 1.$$

Then clearly $x_1,\ldots,x_{2n-1}$ are the first $2n$ zeroes of $U \cdot \varphi$, in their natural order, in $(a,\hat{x})$.

It is easy to check that

$$\Phi(x_j) > 0 \quad \text{for} \quad 1 < j < 2n - 1. \qquad (4.31)$$

Indeed, for odd j, this follows from (4.19.a) and (4.29). For even j, say $j = 2k$, this due to the sign of U in I_{k+1} and the sign of φ' at s_k: both are $(-1)^k$.

Next, set $p(x) = w'(x)/w(x)$ for $x > a$. Since $U \cdot \varphi$ and consequently Φ' does not change sign in (x_{j-1}, x_j), by the Mean Value Theorem there exists $t_j \in (x_{j-1}, x_j)$ such that

$$\int_{x_{j-1}}^{x_j} p(x)\Phi'(x)dx = p(t_j) \int_{x_{j-1}}^{x_j} \Phi'(x)dx, \quad 1 \le j \le 2n \ . \tag{4.32}$$

From (4.20), $p\Phi' = -\sigma\Psi'$, which, together with (4.32), leads to

$$\sigma[\Psi(x_j) - \Psi(x_{j-1})] = -p(t_j)[\Phi(x_j) - \Phi(x_{j-1})], \quad 1 \le j \le 2n \ . \tag{4.33}$$

Summing up from $j = 1$ to $2n$ yields

$$\sigma[\Psi(\hat{x}) - \Psi(a)] = -\sum_{j=1}^{2n} p(t_j)[\Phi(x_j) - \Phi(x_{j-1})]$$

$$= -\Phi(\hat{x})p(t_{2n}) + \Phi(a)p(t_1) + \sum_{j=1}^{2n-1} \Phi(x_j)[p(t_{j+1})-p(t_j)] \ .$$

By hypothesis (w.1) and (4.31), (4.21.a)

$$\sigma(\Psi(\hat{x}) - \Psi(a)] > -\Phi(\hat{x})p(t_{2n}) \ .$$

Finally, to obtain (4.30), we use (4.33) to eliminate $p(t_{2n})$ from the last inequality. We can do so since Φ' does not vanish in $(x_{2n-1};\hat{x})$.

<u>Lemma 4.34</u>

Assume (r.2), (F.6) and (ψ.3) are satisfied.

Let $\bar{w} = \min_{x \in [a,b]} w(x)$ and $c = \mathrm{Max}((\psi_1^{-1}(\lambda)/\bar{w})^{1/\sigma}, (\psi_2^{-1}(\lambda)/\bar{w})^{1/\sigma})$.

If v is a solution of $(I)_{a,b}$ then

$$\|v\|_{L^\infty[a,b]} < c \tag{4.35.a}$$

and

$$\|v'\|_{L^\infty[a,b]} < \sqrt{\lambda} \cdot c \ . \tag{4.35.b}$$

Proof

Suppose v attains a positive maximum at $s \in (a,b)$. Then $v''(s)/v(s) < 0$. Now due to the hypotheses, equation (1.4.a) should be read as (4.5) which implies that

$$\lambda - \psi_1(w(s)|v(s)|^{\sigma}) > 0$$

and hence

$$v(s) < (\psi_1^{-1}(\lambda)/w(s))^{1/\sigma} < (\psi_1^{-1}(\lambda)/\bar{w})^{1/\sigma} . \qquad (4.36)$$

In view of the boundary conditions (1.4.b), we know the only way that v could attain its maximum at a boundary point occurs at $x = a$ and when $\theta = \frac{\pi}{2}$. In this case $u'(a) = 0$ and $u''(a) < 0$. Thus, the same argument as above shows (4.36) holds at $s = a$. Therefore, we have

$$v(x) < (\psi_1^{-1}(\lambda)/\bar{w})^{1/\sigma} \quad \text{for all} \quad x \in [a,b] .$$

A similar argument for negative minima of v yields

$$v(x) > -(\psi_2^{-1}(\lambda)/\bar{w})^{1/\sigma} \quad \text{for all} \quad x \in [a,b] .$$

Thus (4.35.a) follows. Combining (1.65) with (4.35.a), we get (4.35.b).

For the next result, we suppress the dependence of a and θ from our notation.

Proposition 4.37

Suppose (r.1) and (f.2) are satisfied. Let $a > 0$ and $0 < \theta < \frac{\pi}{2}$ be fixed. Let $n \in \mathbb{N}$ and $(\lambda_k,\xi_k) \in \mathcal{D}_n^+$ (resp. $\mathcal{D}_n^-$) such that $\lim_{k\to\infty} (\lambda_k,\xi_k) = (\lambda,\xi) \in \mathcal{D}_0^+ \cap \partial\mathcal{D}_n^+$ (resp. $\mathcal{D}_0^- \cap \partial\mathcal{D}_n^-$) then

$$\lim_{k\to\infty} z_n(\lambda_k,\xi_k) = +\infty.$$

Proof

Suppose this were false, there would exist a $b \in (a,\infty)$ such that, by passing to a subsequence if necessary, $z_n(\lambda_k,\xi_k)$ converges to b as $k \to \infty$. By the continuous dependence of the solution on parameters

$$U(\lambda_k,\xi_k,\cdot) \xrightarrow{C^1[a,b]} U(\lambda,\xi,\cdot) .$$

Since $\xi \neq 0$, $U(\lambda,\xi,\cdot)$ cannot be the trivial solution and hence can only have simple zeroes. Therefore $U(\lambda,\xi,\cdot)$ must have at least n zeroes in (a,∞). This implies $(\lambda,\xi) \in \mathcal{D}_n^+$ (resp. $\mathcal{D}_n^-$) which contradicts that $\mathcal{D}_n^+$ (resp. $\mathcal{D}_n^-$) is open.

Proposition 4.38

Suppose (r.1), (f.2) and (F.3) are satisfied. Let $a > 0$ and $0 < \theta < \frac{\pi}{2}$ be fixed. Let $[\underline{\lambda},\bar{\lambda}]$ be a compact subinterval of $(0,\infty)$. Let $b_k = b_k(\lambda) \in R$ be such that $\mu_k(a,a + b_k,\theta) = \lambda$. Then, for every $n \in N$, if $b > b_n(\underline{\lambda})$, there exists a positive number $\varepsilon_n = \varepsilon_n(\underline{\lambda},\bar{\lambda},b)$ such that if $0 < |\xi| < \varepsilon_n$ and $\lambda \in [\underline{\lambda},\bar{\lambda}]$, $U_a(\lambda,\xi,\cdot)$ has at least n zeroes in (a,b). Moreover

$$\lim_{\xi\to 0} z_{a,n}(\lambda,\xi) = a + b_n(\lambda) .$$

Proof

Let $v_{\lambda,\xi} = \xi^{-1} \circ U_a(\lambda,\xi,\cdot)$ for $\xi > 0$, then $v_{\lambda,\xi}$ is the solution of the initial value problem

$$-v'' = [\lambda r(x) - F(x,U_a(\lambda,\xi,x))]v , \tag{4.40.a}$$

$$\begin{aligned} &v(a) = 0, \quad v'(a) = 1 \quad \text{if } \theta = 0 ,\\ &v(a) = 1, \quad v'(a) = \cot\theta \quad \text{if } 0 < \theta < \frac{\pi}{2} . \end{aligned} \tag{4.40.b}$$

Next, let v_λ be the solution of differential equation

$$-v'' = \lambda r(x)v$$

together with initial conditions (4.40.b). Then it follows from the assumption (F.3) and the basic theory of initial value problems [35] that

$$\lim_{\xi \to 0} v_{\lambda,\xi} = v_\lambda$$

and

$$\lim_{\xi \to 0} v'_{\lambda,\xi} = v'_\lambda$$

uniformly on compact subsets of $\{(\lambda,x) | \lambda > 0,\ x > a\}$. Pick $b > b_n(\underline{\lambda})$. Since $\mu_n(a,\cdot,\theta)$ is a decreasing function, $b > b_n(\lambda)$ for $\lambda \in [\underline{\lambda},\bar{\lambda}]$. Thus, there is an $\varepsilon_n = \varepsilon_n(\underline{\lambda},\bar{\lambda},b)$ such that for $0 < \xi < \varepsilon_n$ and $\lambda \in [\underline{\lambda},\bar{\lambda}]$ the function $v_{\lambda,\xi}$ is defined and has at least n zeroes in (a,b). Moreover, the first n zeroes of $v_{\lambda,\xi}$ tend to those of v_λ as $\xi \to 0$.

The case $-\varepsilon_n < \xi < 0$ can be treated similarly.

Now we are ready to verify Proposition 4.12.

<u>Proof of Proposition 4.12</u>

We only verify the result for D_n^+. Recall that φ was defined in (4.16). Suppose, for every $\xi \in D_n^+$

$$\varphi(\xi,z_n(\xi))/U'(\xi,z_n(\xi)) < 0 \tag{4.41}$$

then the equation $U(\xi,z_n(\xi)) = 0$ yields

$$\frac{dz_n(\xi)}{d\xi} = -\frac{\varphi(\xi,z_n(\xi))}{U'(\xi,z_n(\xi))} \tag{4.42}$$

and hence (4.13) immediately follows from (4.41).

Next, we prove (4.41) by induction on n. By (4.17.b) and Lemma 4.22 in case of $\theta = 0$, or Lemma 4.26 otherwise, we know

$\varphi(\xi, z_1(\xi)) > 0$. This together with $U'(\xi, z_1(\xi)) < 0$ implies (4.41) holds for $n = 1$.

Suppose, for $n \geq 2$, that (4.41) is established for all indices up to $n - 1$ and suppose there exists a $\xi_0 \in D_n^+$ such that $\varphi(\xi_0, z_n(\xi_0)) = 0$. Then, by Lemma 4.22, $\varphi(\xi_0, x) \neq 0$ for $x \subset (z_{n-1}(\xi_0), z_n(\xi_0))$. Using the induction hypothesis, (4.29) holds for $1 \leq k \leq n - 1$. Thus applying Proposition 4.28, we get

$$\Psi(\hat{x}) \geq \Psi(a) + \phi(\hat{x}) \frac{\Psi(\hat{x}) - \Psi(z_{n-1})}{\Phi(\hat{x}) - \Phi(z_{n-1})} .$$

Letting $\hat{x} = z_n(\xi_0)$ and using the fact that $\phi(\xi_0, z_n(\xi_0)) = 0$ we obtain

$$\Psi(\xi_0, z_n(\xi_0)) \geq \Psi(\xi_0, a) . \tag{4.43}$$

In case of $\theta = 0$, by (4.21.b)

$$\Psi(\xi_0, a) = \xi_0 > 0 .$$

If $0 < \theta \leq \frac{\pi}{2}$ by (4.21.b)

$$\Psi(\xi_0, a) = \xi_0 \cot^2\theta - U''(\xi, a) .$$

Since $w'(a) \geq 0$ and $w(x) > 0$ for $x \geq a$, by hypothesis (w.1), $w'(x) \geq 0$ for $x > a$. Thus applying Lemma 4.23, we have $U''(\xi, a) < 0$ and consequently

$$\Psi(\xi_0, a) > 0 .$$

Thus, in all situations $0 \leq \theta \leq \frac{\pi}{2}$, $\Psi(\xi_0, a) > 0$ and hence, by (4.43),

$$\Psi(\xi_0, z_n(\xi_0)) > 0 . \tag{4.44}$$

On the other hand, we know $z_n(\xi_0)$ is the n-th zero of $U(\xi_0, \cdot)$ and the $(n - 1)$-th zero of $\varphi(\xi_0, \cdot)$ in (a, ∞). By (4.1.b) and

(4.17.b), it is easy to check that $\Psi(\xi_0, z_n(\xi_0)) = \varphi'(\xi_0, z_n(\xi_0)) \cdot U'(\xi_0, z_n(\xi_0))$ is negative. But this is contrary to (4.44). Thus, there is no $\xi_0 \in D_n^+$ such that $\varphi(\xi_0, z_n(\xi_0)) = 0$ and consequently $\varphi(\xi, z_n(\xi))/U'(\xi, z_n(\xi))$ is of constant sign on each connected component of D_n^+.

Suppose there is a nonempty connected component C of D_n^+ on which the sign is positive. Then it follows from (4.42) that the function z_n is monotonically decreasing on C and hence choosing $\xi_1 \in C$ and setting $b = z_n(\xi_1)$ we have $z_n(\xi) < b$ for every $\xi \in C \cap [\xi_1, \infty)$. Since $C \cap [\xi_1, \infty)$ is a nonempty connected open set it must be an interval $[\xi_1, \xi_2)$, $\xi_2 < +\infty$. Suppose $\xi_2 < +\infty$. By Proposition 4.37

$$\lim_{\xi \to \xi_2} z_n(\xi) = +\infty$$

which is contrary to that $z_n(\xi) < b$ for all $\xi \in [\xi_1, \xi_2)$. Hence $\xi_2 = +\infty$ and $[\xi_1, \infty) \subset C$. Let $v_\xi(x) = U(\xi, x)$ for $x \in [a, z_n(\xi)]$. Then, by Lemma 4.34 and $z_n(\xi) < b$ for $\xi \in [\xi_1, \infty)$, $\|v_\xi'\|_{L^\infty[a, z_n(\xi)]}$ are uniformly bounded. This contradicts to $v_\xi'(a) = \xi$. Therefore (4.41) holds on all of D_n^+.

Now we are going to show D_n^+ contains only one connected component. Suppose (η_1, η_2) is a connected component of D_n^+ by (4.13) we know that

$$\lim_{\xi \to \eta_1} z_n(\xi) = \operatorname*{Inf}_{(\eta_1, \eta_2)} z_n(\xi) .$$

On the other hand, if $\eta_1 \neq 0$ it follows from Proposition 4.37 that

$$\lim_{\xi \to \eta_1} z_n(\xi) = +\infty .$$

Thus, η_1 must be zero, otherwise we would have a contradiction. Since (η_1, η_2) is an arbitrary connected component, we conclude that D_n^+ contains only one connected component and has the form $(0, \Omega_n^+)$ where $0 < \Omega_n^+ \leq +\infty$.

Finally, if $\Omega_n^+ < +\infty$ we obtain (4.14) from Proposition 4.37. If $\Omega_n^+ = +\infty$ and (4.14) were false, then there would exist $b > a$ such that

$$\lim_{\xi \to \infty} z_n(\xi) = \sup_{(0,\infty)} z_n(\xi) \leq b .$$

By letting $v_\xi(x) = U(\xi, x)$ for $x \in [a, z_n(\xi)]$ and using Lemma 4.34 it would lead a contradiction as before. This completes the proof.

Remark 4.45

From the above proof we know if $\xi \in D_n^+$ or D_n^-, (4.18) implies $\varphi(\xi, z_k(\xi)) \cdot U'(\xi, z_k(\xi)) < 0$ for $1 \leq k \leq n$. Thus the zeroes of $\varphi(\xi, \cdot)$ and those of $U(\xi, \cdot)$ are interlaced.

Proof of Theorem 4.6

By (4.13), we know S_n^+ as well as S_n^- contains at most one element. However, Proposition 1.43 and Remark 1.44 indicate that S_n^+ and S_n^- are nonempty. Thus the result follows.

Since the proof of Theorem 4.7 needs more preliminaries and Theorem 4.10 is closely related to Theorem 4.6, we prove Theorem 4.10 first.

Proof of Theorem 4.10

Let

$$T(\mu,v) = \begin{cases} v'' + \mu v - \psi_1(w|v|^{\sigma})v, & \text{if } v > 0 \\ v'' + \mu v - \psi_2(w|v|^{\sigma})v, & \text{if } v < 0 . \end{cases}$$

Then $T : R \times C_\theta^2[a,b] \to C[a,b]$ where $C_\theta^2 = \{g \in C^2[a,b] \mid g(a)\cos\theta - g'(a)\sin\theta = 0,\ g(b) = 0\}$. Let (λ,u) be a zero of T with $u \in S_{a,b,n}^+(\lambda,\theta)$. If $T_v(\lambda,u)$, the Fréchet differential of T with respect to v, is an isomorphism, then the hypotheses of the implicit function theorem [30] are satisfied and hence there is an $\varepsilon > 0$ and a C^1-mapping $\mu \to u_n^+(\mu)$ for $|\mu - \lambda| < \varepsilon$ such that $T(\mu,u_n^+(\mu)) = 0$. Thus, it suffices to show $T_v(\lambda,u)$ is an isomorphism for every pair (λ,u) such that $u \in S_{a,b,n}^+(\lambda,\theta)$ or equivalently to show that 0 is not an eigenvalue of $T_v(\lambda,u)$ [35]:

$$T_v(\lambda,u)\varphi_1 = \begin{cases} \varphi_1''+[\lambda-\psi_1(w|u|^{\sigma})-\sigma\psi_1'(w|u|^{\sigma})w|u|^{\sigma}]\varphi_1, & \text{if } u > 0 \\ \varphi_1''+[\lambda-\psi_2(w|u|^{\sigma})-\sigma\psi_2'(w|u|^{\sigma})w|u|^{\sigma}]\varphi_1, & \text{if } u < 0 , \end{cases} \quad (4.46.a)$$

$$\varphi_1(a)\cos\theta - \varphi_1'(a)\sin\theta = 0, \qquad \varphi(b) = 0 . \quad (4.46.b)$$

Let $u \in S_{a,b,n}^+(\lambda,\theta)$. Let $\xi = u'(a)$ if $\theta = 0$ and $\xi = u(a)$ if $0 < \theta < \frac{\pi}{2}$ then $U_a(\lambda,\xi,\theta,x) = u(x)$ for $x \in [a,b]$ and $\tau_{a,n}(\lambda,\xi,\theta) = b$. Suppose 0 is an eigenvalue of $T_v(\lambda,u)$ with eigenfunction φ_1. Multiplying by a constant if necessary, φ_1 satisfies (4.17.b). By the basic uniqueness result of initial value problems $\varphi_1 = \frac{\partial}{\partial\xi} U_a(\lambda,\xi,\theta,\cdot) = \varphi(\xi,\cdot)$. But from Remark 4.45, we know $\varphi(\xi,b) \neq 0$ which is contrary to (4.46.b). Thus we complete the proof.

Now, we continue with the preliminary work needed for the proof of Theorem 4.7. Let $\lambda > 0$ and $0 < \theta < \frac{\pi}{2}$. For $n > 1$, we define

$$Q^+_{a,n}(\lambda,\theta) \quad (\text{resp. } Q^-_{a,n}(\lambda,\theta)) = \{\xi \mid U_a(\lambda,\xi,\theta,\cdot) \in S^+_{a,n}(\lambda,\theta) \quad (\text{resp. } S^-_{a,n}(\lambda,\theta)\} .$$

Again, for convenience, we will suppress dependence on λ, θ or a from our notation whenever it is considered fixed.

Proposition 4.47

Let $a > 0$, $\lambda > 0$ be fixed. Under the same hypotheses as in Theorem 4.7, we have

(i) $Q^+_1 = \{\Omega^+_1\}$, $Q^-_1 = \{\Omega^-_1\}$.

(ii) For $n > 2$, $Q^+_n \subset [\Omega^+_n, \Omega^+_{n-1})$ (resp. $(\Omega^-_{n-1}, \Omega^-_n]$).

(iii) If Q^+_n (resp. Q^-_n) $\neq \emptyset$ then $\Omega^+_n \in Q^+_n$ (resp. $\Omega^-_n \in Q^-_n$).

(iv) If Q^+_n (resp. Q^-_n) contains a unique element then $Q^+_n = \{\Omega^+_n\}$ (resp. $Q^-_n = \{\Omega^-_n\}$).

Proof

Let $n > 1$. Pick an increasing sequence $\{\xi_k\}$ such that

$$\lim_{k\to\infty} \xi_k = \Omega^+_n .$$

Put

$$v_k(x) = \begin{cases} U(\xi_k, x) & \text{if } x \in [a, z_n(\xi_k)] \\ 0 & \text{if } x \in [z_n(\xi_k), \infty) . \end{cases}$$

By Proposition 4.37

$$\lim_{k\to\infty} z_n(\xi_k) = +\infty . \tag{4.48}$$

From the Remark 4.4, we know the nypothesis of Theorem 1.2 are satisfied. Hence, by Lemma 1.57 and arguing like the proof of Theorem 1.2 we have a prior bounds on v_k and obtain a subsequence $\{v_{k_\ell}\}$ and a $v \in C^2[a,\infty) \cap H^1[a,\infty)$ such that

$$v_{k_\ell} \xrightarrow{C^2} v \text{ uniformly on compact subintervals of } [a,\infty)$$

for $x \in [a,\infty)$. Thus, by the uniqueness result for the initial value problem, $v = U(\Omega_n^+,\cdot)$. Also note that, from (4.48), $U(\Omega_n^+,\cdot)$ has at most $n - 1$ zeroes in (a,∞).

For $n = 1$, it is clear that $v \in S_1^+$. Hence, by Theorem 1.5, (i) follows.

For $n \geq 2$, if $Q_n^+ = \emptyset$ there is nothing to prove in (ii). If $Q_n^+ \neq \emptyset$, since for any $\xi \in Q_n^+$, $\xi \in D_{n-1}^+ \backslash D_n^+$ and by Proposition 4.12, $D_{n-1}^+ \backslash D_n^+ = [\Omega_n^+, \Omega_{n-1}^+)$, we have (ii).

To prove (iii), let $\eta \in Q_n^+$, it follows from (ii) that $\eta \geq \Omega_n^+$. Suppose $\eta > \Omega_n^+$. Since $U(\eta,\cdot)$ has $n - 1$ zeroes in (a,∞), by Proposition 4.12, $U(\Omega_n^+,\cdot)$ has at least $n - 1$ zeroes in (a,∞). However, we already know $U(\Omega_n^+,\cdot) \in C^2[a,\infty) \cap H^1[a,\infty)$ and has at most $n - 1$ zeroes in $[a,\infty)$. Therefore $\Omega_n^+ \in Q_n^+$.

Finally (iv) simply follows from (iii).

Proposition 4.49

Let $\lambda > 0$ and $a \geq 0$ be fixed. Under the same hypotheses as in Theorem 4.7, if $n \geq 2$ then Q_n^+ (resp. Q_n^-) has no cluster point in $[\Omega_n^+, \Omega_{n-1}^+)$ (resp. $(\Omega_{n-1}^-, \Omega_n^-]$).

The proof will be carried out only the case of Q_n^+. Some preliminaries are needed.

Lemma 4.50

Let $\lambda > 0$, $a \geq 0$ be fixed. Assume the hypotheses of Theorem 4.7 are satisfied. Suppose $\xi \in Q_n^+$ for some $n \geq 2$ and suppose that

$$U(\xi,x)\varphi(\xi,x) < 0 \qquad (4.51)$$

for $x \in (z_{n-1}(\xi),\infty)$. Then there exists a $t \in (z_{n-1}(\xi),\infty)$ such that

$$\Psi(\xi,t) > 0 . \tag{4.52}$$

Proof

From Remark 4.45, we know (4.29) holds for $1 \leq k \leq n-1$. Hence, it follows from (4.30) that

$$\Psi(x) \geq \Psi(a) + \Phi(x)\,\frac{\Psi(x) - \Psi(z_{n-1}(\xi))}{\Phi(x) - \Phi(z_{n-1}(\xi))} \tag{4.53}$$

for $x \in (z_{n-1}(\xi),\infty)$. Let

$$\bar{\xi} = \mathrm{Inf}\{x \mid w'(x) > 0\} . \tag{4.54}$$

By assumption (w.2), we know $\bar{\xi} < +\infty$.

Suppose $\bar{\xi} > z_{n-1}(\xi)$. The assumption (4.51) together with (4.20) implies $\Phi'(x) < 0$ for $x \in (z_{n-1}(\xi),\bar{\xi})$. So

$$\Phi(x) < \Phi(z_{n-1}(\xi)) . \tag{4.55}$$

Similarly, $\Psi'(x) \leq 0$ for $x \in (z_{n-1}(\xi),\bar{\xi})$. Hence

$$\Psi(x) \leq \Psi(z_{n-1}(\xi)) . \tag{4.56}$$

By Remark 4.45 with $k = n-1$, (4.19.a) yields $\Phi(z_{n-1}(\xi)) > 0$. Hence, there exists a $\delta_1 > 0$ such that $\Phi(x) > 0$ for $x \in [z_{n-1}(\xi), z_{n-1}(\xi) + \delta_1]$. Combining this with (4.54)-(4.56), we obtain

$$\Psi(x) \geq \Psi(a) .$$

From (4.21.b) and Lemma 4.23,

$$\Psi(a) > 0 . \tag{4.57}$$

Therefore, $\Psi(x) > 0$ for $x \in [z_{n-1}(\xi), z_{n-1}(\xi) + \delta_1]$.

If $\bar{\xi} \leq z_{n-1}(\xi)$, by (4.20.b) and (4.51), $\Psi'(x) > 0$ in $(z_{n-1}(\xi),\infty)$. Hence, by letting $L_1 = \lim_{x\to\infty} \Psi(x)$, where $-\infty < L_1 \leq +\infty$ we have

$$L_1 - \Psi(x) > 0 \quad \text{for} \quad x \in [z_{n-1}(\xi),\infty) \tag{4.58}$$

and (4.52) easily follows if $L_1 > 0$. To show $L_1 > 0$. Note that from (4.20.a) and (4.51), $\Phi'(x) < 0$ in $(z_{n-1}(\xi),\infty)$. Thus, by letting $L_2 = \lim\limits_{x\to\infty} \Phi(x)$ where $-\infty \leq L_2 < +\infty$, we have

$$L_2 - \Phi(x) < 0 \quad \text{for} \quad x \in [z_{n-1}(\xi),\infty) \ . \tag{4.59}$$

Letting $x \to +\infty$ in (4.53) yields

$$L_1 \geq \Psi(a) + L_3(L_1 - \Psi(z_{n-1}(\xi))) \tag{4.60}$$

where

$$L_3 = \begin{cases} 1 & \text{if} \quad L_2 = -\infty \\ L_2/(L_2 - \Phi(z_{n-1}(\xi))) & \text{if} \quad L_2 > -\infty \ . \end{cases}$$

Suppose $L_2 \leq 0$. By (4.59), we get $L_3 \geq 0$. Combining this with (4.58) and (4.60), we obtain $L_1 \geq \Psi(a)$. By (4.57), $L_1 > 0$. Therefore it remains to show $L_2 \leq 0$. We argue indirectly. Note that, by Lemma 1.7, we know

$$\lim_{x\to\infty} U(x) = 0 \ .$$

Since $(\varphi/U)' = \Phi/U^2 \to +\infty$ as $x \to +\infty$, if $L_2 > 0$, we have $(\varphi/U) \to +\infty$ as $x \to +\infty$. But this violates (4.51). So we must have $L_2 \leq 0$. This completes the proof.

<u>Lemma 4.61</u>

Assume the hypotheses of Proposition 4.12 are satisfied. Let $\xi,\eta \in \Omega_n^+$ for some $n \geq 2$ and $\xi < \eta$. Then for $x \geq z_{n-1}(\eta)$, we have

$$|U(\xi,x)| > |U(\eta,x)| \ .$$

Proof

From Proposition 4.12, we have $z_{n-1}(\eta) > z_{n-1}(\xi)$. Next, by Theorem 1.5, $U(\xi,x) = V_{\tau(n)}(\lambda, z_{n-1}(\xi), \infty, 0, x)$ for $x \in [z_{n-1}(\xi), \infty)$ and $U(\eta,x) = V_{\tau(n)}(\lambda, z_{n-1}(\eta), \infty, 0, x)$ for $x \in [z_{n-1}(\eta), \infty)$. (Recall the function τ was defined in (2.53).) Invoking (1.50), we have

$$|V_{\tau(n)}(\lambda, z_{n-1}(\xi), \infty, 0, x)| > |V_{\tau(n)}(\lambda, z_{n-1}(\eta), \infty, 0, x)|$$

for $x \in [z_{n-1}(\eta), \infty)$ and thus the result follows.

Remark 4.62

In this section, since $f(x,y)$ is assumed to be continuously differentiable, the hypothesis (f.1) is clearly satisfied. Hence, as pointed out there, the inequality (1.50) as well as others should be interpreted in the strict sense. In the remainder of this section, we will use this fact without further comment.

Proof of Proposition 4.49

Let us define the function $E = E(\xi,x)$ by

$$E(\xi,x) = \begin{cases} \frac{1}{2} u'^2(x) + \frac{\lambda}{2} u^2(x) - \int_0^{u(x)} \psi_1(w(x)|y|^\sigma) y\,dy & \text{if } u(x) > 0 \\ \frac{1}{2} u'^2(x) + \frac{\lambda}{2} u^2(x) - \int_0^{u(x)} \psi_2(w(x)|y|^\sigma) y\,dy & \text{if } u(x) < 0 \end{cases}$$

where $u = U(\xi,\cdot)$. Then

$$\frac{\partial E}{\partial x} = \begin{cases} u'u'' + \lambda uu' - \psi_1(w|u|^\sigma) uu' - \int_0^{u(x)} \psi_1'(w|y|^\sigma) w' |y|^\sigma y\,dy & \text{if } u(x) > 0 \\ u'u'' + \lambda uu' - \psi_2(w|u|^\sigma) uu' - \int_0^{u(x)} \psi_2'(w|y|^\sigma) w' |y|^\sigma y\,dy & \text{if } u(x) < 0. \end{cases}$$

Substituting (4.5) into the last equation, we obtain

$$\frac{\partial E}{\partial x} = \begin{cases} -\int_0^{u(x)} \psi_1'(w|y|^\sigma)w'|y|^\sigma y\,dy & \text{if} \quad u(x) > 0 \\ -\int_0^{u(x)} \psi_2'(w|y|^\sigma)w'|y|^\sigma y\,dy & \text{if} \quad u(x) < 0 . \end{cases} \tag{4.63}$$

Next, letting $\varphi = \varphi(\xi,\cdot)$, then

$$\frac{\partial E}{\partial \xi} = \begin{cases} u'\varphi' + \lambda u\varphi - \psi_1(w|u|^\sigma)u\varphi & \text{if} \quad u(x) > 0 \\ u'\varphi' + \lambda u\varphi - \psi_2(w|u|^\sigma)u\varphi & \text{if} \quad u(x) < 0 . \end{cases}$$

Combining (4.5) and (4.19.b) with the last equation, it follows that

$$\frac{\partial E}{\partial \xi} = \Psi(\xi,x) . \tag{4.64}$$

If $\xi \in Q_n^+$, $u = U(\xi,\cdot) \in H^1[a,\infty)$. This together with Lemma 2.12 implies $E(\xi,\cdot) \in L^1[a,\infty)$. Let $s = \text{Max}(\hat{b}, z_{n-1}(\xi))$ where $\hat{b}$ was defined in (w.2). We claim that $E(\xi,\cdot)$ is decreasing on (s,∞) and therefore, we have

$$E(\xi,x) \to 0 \quad \text{as} \quad x \to \infty . \tag{4.65}$$

Indeed, if $U(\xi,x) > 0$ for $x \in (z_{n-1}(\xi),\infty)$ (4.63) together with (w.1) and (w.2) shows that $E(\xi,\cdot)$ is decreasing for $x \in (s,\infty)$. If $U(\xi,x) < 0$ for $x \in (z_{n-1}(\xi),\infty)$, (4.63) yields

$$\frac{\partial E}{\partial x} = \int_{u(x)}^{0} \psi_2'(w(x)|y|^\sigma)w'|y|^\sigma y\,dy .$$

Combining with (w.1) and (w.2) again shows that $E(\xi,\cdot)$ is decreasing on (s,∞).

Consider those $\xi \in Q_n^+$ such that there is a decreasing sequence $\{\xi_k\}$ in Q_n^+ which converges to ξ as $k \to \infty$. Then, by (4.13), $z_{n-1}(\xi)$ is the limit of the decreasing sequence $\{z_{n-1}(\xi_k)\}$. Hence, for every $x > z_{n-1}(\xi)$, we have $x > z_{n-1}(\xi_k)$ provided k is sufficiently large. From Lemma 4.61 we know

$$|U(\xi,x)| > |U(\xi_k,x)| \tag{4.66}$$

for $x \in (z_{n-1}(\xi_k),\infty)$.

If $U(\xi,x) > 0$ for $x \in (z_{n-1}(\xi),\infty)$ then $U(\xi_k,x) > 0$ for $x \in (z_{n-1}(\xi_k),\infty)$. Thus, by (4.63) and (4.65), we have

$$E(\xi,x) = \int_x^\infty \int_0^{U(\xi,t)} \psi_1'(w(t)|y|^\sigma)w'(t)|y|^\sigma y\,dy\,dt$$

for $x \in (z_{n-1}(\xi),\infty)$ and

$$E(\xi_k,x) = \int_x^\infty \int_0^{U(\xi_k,t)} \psi_1'(w(t)|y|^\sigma)w'(t)|y|^\sigma y\,dy\,dt$$

for $x \in (z_{n-1}(\xi_k),\infty)$. Combining this with (4.66), we get, for any $x > z_{n-1}(\xi)$,

$$E(\xi_k,x) < E(\xi,x)$$

provided k is large enough.

Since $\frac{\partial E}{\partial \xi}$ is known to exist the last inequality together with (4.64) implies

$$\Psi(\xi,x) \leq 0 \tag{4.67.a}$$

for every $x > z_{n-1}(\xi)$. However, (4.66) and (4.16) also show that $\varphi(\xi,x)$ is nonpositive on $[z_{n-1}(\xi),\infty)$. Since $\varphi(\xi,\cdot)$ is a solution of (4.17) it has only simple zeroes. Therefore

$$\varphi(\xi,x) < 0 \quad \text{for} \quad x \in (z_{n-1}(\xi),\infty) \tag{4.67.b}$$

and hence

$$U(\xi,\cdot)\varphi(\xi,\cdot) < 0$$

in $(z_{n-1}(\xi),\infty)$. By Lemma 4.50, there exists an $x \in (z_{n-1}(\xi),\infty)$ such that $\Psi(\xi,x) > 0$ which is contrary to (4.67).

If $U(\xi,x) < 0$ for $x \in (z_{n-1}(\xi),\infty)$, the same line of reasoning with only the sign of (4.67.b) reversed yields a contradiction to (4.67). Therefore, if $\xi \in Q_n^+$ it cannot be the limit of a decreasing sequence $\{\xi_k\} \subset Q_n^+$. Likewise, with slight modifications in the above argument it cannot be the limit of an increasing sequence in Q_n^+. Since any convergent sequence contains a monotone subsequence, the proof is completed.

Recall the notations $V_\pm(\lambda,a,b,\theta,x)$ and $\mu_n(a,b,\theta)$ which were defined in Remark 1.6 (b) and (1.41) respectively.

<u>Lemma 4.68</u>

Assume (r.1), (F.1)-(F.4) and (f.1) are satisfied. Let $b \in (a,\infty)$ and $\lambda > \mu_1(a,b,\theta)$. Then, for $x \in [a,b]$

$$V_\pm'(\lambda,a,b,\theta,x) = U_a'(\lambda,V_\pm'(\lambda,a,b,\theta,a),\theta,x) \quad \text{if} \quad \theta = 0 ,$$

$$V_\pm'(\lambda,a,b,\theta,x) = U_a'(\lambda,V_\pm(\lambda,a,b,\theta,a),\theta,x) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2} .$$

<u>Proof</u>

It simply follows from Theorem 1.5, the definition of $V_\pm(\lambda,a,b,\theta,\cdot)$ and that of $U_a(\lambda,\xi,\theta,\cdot)$.

<u>Lemma 4.69</u>

Assume the hypotheses of Proposition 4.12 are satisfied. Suppose $\lambda > \mu_1(a,b,\theta)$. Let $\xi = V_+'(\lambda,a,b,0,a)$ (resp. V_-') if $\theta = 0$ or $\xi = V_+(\lambda,a,b,\theta,a)$ (resp. V_-) if $0 < \theta < \frac{\pi}{2}$. Let $\eta = V_+'(\lambda,a,b,\theta,b)$ (resp. V_-')

(i) If $\eta < \Omega_{b,n}^-(0)$ (resp. $> \Omega_{b,n}^+(0)$) then $\xi > \Omega_{a,n+1}^+(\theta)$ (resp. $< \Omega_{a,n+1}^-(\theta)$).

(ii) If $\eta > \Omega_{b,n}^-(0)$ (resp. $< \Omega_{b,n}^+(0)$) then $\xi < \Omega_{a,n+1}^+(\theta)$ (resp. $> \Omega_{a,n+1}^-(\theta)$).

Proof

(i) If $\eta < \Omega^-_{b,n}(0)$ by Proposition 4.12, $\eta \notin D^-_{b,n}(0)$. Hence $\xi \notin D^+_{a,n+1}(\theta)$. Thus, by Proposition 4.12, we know $\xi > \Omega^+_{a,n+1}(\theta)$.

(ii) If $\eta > \Omega^-_{b,n}(0)$, by Proposition 4.12 $\eta \in D^-_{b,n}(0)$. Hence $\xi \in D^+_{a,n+1}(\theta)$. Thus, by Proposition 4.12, we have $\xi < \Omega^+_{a,n+1}(\theta)$.

Proposition 4.70

Assume the hypotheses of Theorem 4.7 are satisfied. Let $\lambda > 0$, $0 \leq \theta \leq \frac{\pi}{2}$ be fixed and $n \geq 2$. If $\alpha > a \geq 0$ and $\mathcal{Q}^+_{\alpha,n}(\theta)$ (resp. $\mathcal{Q}^-_{\alpha,n}(\theta)) \neq \varphi$, then $\mathcal{Q}^+_{a,n}(\theta)$ (resp. $\mathcal{Q}^-_{a,n}(\theta)) \neq \varphi$.

Proof

We suppress λ from the notations $V_\pm$, U_α, and etc. Let $\xi \in \mathcal{Q}^+_{\alpha,n}(\theta)$ and suppose $U_\alpha(\xi,\theta,\cdot)$ has interior zeroes at $x_1, x_2, \ldots, x_{n-1}$. It is clear that

$$U_\alpha(\xi,\theta,x) = \begin{cases} V_+(\alpha,x_1,\theta,x) & \text{if } x \in [\alpha,x_1] \\ V_{\tau(i)}(x_{i-1},x_i,0,x) & \text{if } x \in [x_{i-1},x_i], \quad 2 \leq i \leq n \end{cases}$$

where $x_n = +\infty$ and the function τ was defined in (2.53).

Since $\lambda > \mu_1(\alpha,x_1,\theta) > \mu_1(a,x_1,\theta)$, by Proposition 1.43 $V_+(a,x_1,\theta,\cdot)$ exists. By Corollary 1.45 (ii) if $\theta = 0$ or by Corollary 1.45 (iii) if $0 < \theta \leq \frac{\pi}{2}$, we know

$$|V'_+(a,x_1,\theta,x)| > |V'_+(\alpha,x_1,\theta,x)| .$$

Since

$$V'_+(\alpha,x_1,\theta,x_1) = U'_\alpha(\xi,\theta,x_1) = V'_-(x_1,x_2,0,x_1)$$

we get

$$|V'_+(a,x_1,\theta,x_1)| > |V'_-(x_1,x_2,0,x_1)| . \tag{4.71}$$

Now letting $x_0 = a$ and looking at $V_{\tau(1)}(x_0, x_1, \theta, \cdot)$ and $V_{\tau(i)}(x_{i-1}, x_i, 0, \cdot)$, $2 \leq i \leq n$, with the inequality (4.71), we satisfy the hypotheses of Lemma 2.52. Hence there exists a $v \in S^+_{a,n}(\theta)$. Letting $\eta = v'(a)$ if $\theta = 0$ and letting $\eta = v(a)$ if $0 < \theta < \frac{\pi}{2}$ we have $\eta \in Q^+_{a,n}(\theta)$.

Proof of Theorem 4.7

Let $\lambda > 0$ be fixed. Since for every $0 \leq \theta < \frac{\pi}{2}$ and for every $a > 0$, $Q^+_{a,1}(\theta)$ as well as $Q^-_{a,1}(\theta)$ contains a unique element it is sufficient to prove, by induction, the following statement:

If for every $0 \leq \theta < \frac{\pi}{2}$ and for every $a > 0$, $Q^-_{a,n}(\theta)$ (resp. $Q^+_{a,n}(\theta)$) contains at most one element, then, for every $0 \leq \theta < \frac{\pi}{2}$ and for every $a > 0$, $Q^+_{a,n+1}(\theta)$ (resp. $Q^-_{a,n+1}(\theta)$) cannot have more than one element. (4.72)

Suppose there exist $a > 0$ and $0 \leq \theta < \frac{\pi}{2}$ such that $Q^+_{a,n+1}(\theta)$ contains more than one element. By Propositions 4.47 (ii) and 4.49, there is a $\xi_1 \in Q^+_{a,n+1}(\theta)$ such that $(\Omega^+_{a,n+1}(\theta), \xi_1) \cap Q^+_{a,n+1}(\theta) = \emptyset$. Let $b = z_{a,1}(\Omega^+_{a,n}(\theta), \theta)$ and $s = z_{a,1}(\xi_1, \theta)$. Take any $\alpha \in (a,b)$ such that $\mu_1(\alpha, b, \theta) \prec \lambda$. Then, by Corollary 1.45 (ii) or (iii),

$$V^{\prime}_{+}(a,s,\theta,s) < V^{\prime}_{+}(\alpha,s,\theta,s) .$$

Let $\xi_2 = U^{\prime}_{a}(\xi_1, \theta, s)$. From Lemma 4.68

$$V^{\prime}_{+}(a,s,\theta,s) = \xi_2 .$$

Thus, letting $\xi_3 = V^{\prime}_{+}(\alpha,s,\theta,s)$, we have

$$\xi_2 < \xi_3 .$$

Since $\xi_2 \in Q^-_{s,n}(0)$, by the induction hypotheses and Proposition 4.47

$$\xi_2 = \Omega^-_{s,n}(0) .$$

This yields

$$\xi_3 > \Omega^-_{s,n}(0) \ .$$

By Lemma 4.69 (ii), we have

$$V'_+(\alpha,s,0,\alpha) < \Omega^+_{\alpha,n+1}(0)$$

and (4.73)

$$V_+(\alpha,s,\theta,\alpha) < \Omega^+_{\alpha,n+1}(\theta) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2} \ .$$

Since $Q^-_{s,n}(0) \neq \varphi$, by Proposition 4.70, we know $Q^-_{\beta,n}(0) \neq \varphi$ for $\beta \in [b,s)$. Together with the induction hypothesis and Proposition 4.47 leads to

$$Q^-_{\beta,n}(0) = \{\Omega^-_{\beta,n}(0)\}$$

for $\beta \in [b,s)$.

Now we verify that, for any $\alpha \in (a,b)$ such that $\mu_1(\alpha,b,\theta) < \lambda$ and for any $\beta \in (b,s)$

$$V'_+(\alpha,\beta,\theta,\beta) > \Omega^-_{\beta,n}(0) \ . \tag{4.74}$$

Suppose (4.74) were false, by Lemma 4.69 (i)

$$V'_+(\alpha,\beta,0,\alpha) > \Omega^+_{\alpha,n+1}(0)$$

and

$$V_+(\alpha,\beta,\theta,\alpha) > \Omega^+_{\alpha,n+1}(\theta) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2} \ .$$

Since $\beta < s$, by Corollary 1.45 (i),

$$V'_+(\alpha,\beta,0,\alpha) < V'_+(\alpha,s,0,\alpha)$$

and

$$V_+(\alpha,\beta,\theta,\alpha) < V_+(\alpha,s,\theta,\alpha) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2} \ .$$

Thus,

$$V'_+(\alpha,s,0,\alpha) > \Omega^+_{\alpha,n+1}(0)$$

and

$$V_+(\alpha,s,\theta,\alpha) > \Omega_{\alpha,n+1}(\theta) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2}$$

which would contradict (4.73).

Next, pick a $\beta \in (b,s)$ and let $\alpha \to a$. It follows from (4.74) that

$$\lim_{\alpha\to a} V_+'(\alpha,\beta,\theta,\beta) \geq \Omega_{\beta,n}^-(0) .$$

By Proposition 1.76, we get, for all $\theta \in [0, \frac{\pi}{2}]$,

$$V_+'(a,\beta,\theta,\beta) \geq \Omega_{\beta,n}^-(0) . \tag{4.75}$$

Since $\beta \in (b,s)$, by Corollary 1.45 (i),

$$V_+'(a,b,0,a) < V_+'(a,\beta,0,a) < V_+'(a,s,0,a)$$

and (4.76)

$$V_+(a,b,\theta,a) < V_+(a,\beta,\theta,a) < V_+(a,s,\theta,a) \quad \text{if} \quad 0 < \theta < \frac{\pi}{2} .$$

Let

$$\xi_4 = \begin{cases} V_+'(a,\beta,0,a) & \text{if} \quad \theta = 0 \\ V_+(a,\beta,\theta,a) & \text{if} \quad 0 < \theta < \frac{\pi}{2} . \end{cases}$$

Since $b = z_{a,1}(\Omega_{a,n+1}^+(\theta),\theta)$ and $s = z_{a,1}(\xi_1,\theta)$, (4.76) leads to

$$\Omega_{a,n+1}^+(\theta) < \xi_4 < \xi_1 . \tag{4.77}$$

Suppose equality holds in (4.75). Then it is clear that $\xi_4 \in \mathcal{Q}_{a,n+1}^+(\theta)$. But this together with (4.77) leads a contradiction to the fact that $(\Omega_{a,n+1}^+(\theta),\xi_1) \cap \mathcal{Q}_{a,n+1}^+(\theta) = \varphi$. Therefore we must have

$$V_+'(a,\beta,\theta,\beta) > \Omega_{\beta,n}^-(0) .$$

By Lemma 4.69 (ii), this leads to

$$\xi_4 < \Omega_{a,n+1}^+(\theta)$$

which is contrary to (4.77). Thus we complete the proof.

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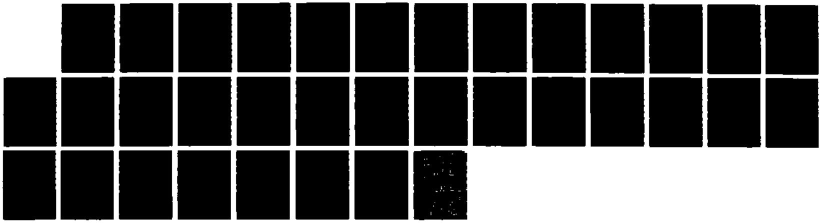

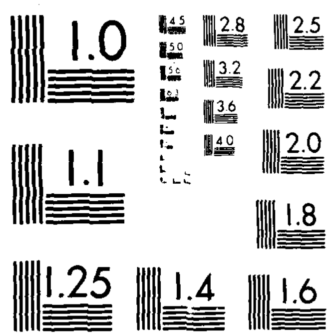

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Now, it remains to prove Theorem 4.11. We first prove two lemmas from which the result is immediate, so as to simplify a later argument in Theorem 5.1.

Lemma 4.78

Assume the hypotheses of Theorem 4.7 are satisfied. Let $a > 0$, $0 < \theta < \frac{\pi}{2}$ be fixed. Then $\Omega_n^{\pm}$ are continuous functions of λ and

$$\lim_{\lambda \to 0^+} \Omega_n^{\pm}(\lambda) = 0 . \tag{4.79}$$

Proof

From Corollary 1.72 and (1.75.a), there exists a continuous function $k(\lambda)$ such that

$$\lim_{\lambda \to 0} k(\lambda) = 0$$

and

$$\Omega_n^{+}(\lambda) < k(\lambda) .$$

Thus (4.79) immediately follows.

Next, for the continuity of $\Omega_n^{+}(\lambda)$. Let $\{\lambda_k\}$ be a sequence such that

$$\lim_{k \to \infty} \lambda_k = \lambda_0 > 0 .$$

Since $\{\lambda_k\}$ is bounded, by Corollary 1.72, $\{\Omega_n^{+}(\lambda)\}$ is bounded. By passing to a subsequence if necessary there is a number L_n such that $\Omega_n^{+}(\lambda_k) \to L_n$ as $k \to \infty$. Since $(\lambda_k, \Omega_n^{+}(\lambda_k)) \in \partial\mathcal{D}_n^{+}$, $(\lambda_0, L_n) \in \partial\mathcal{D}_n^{+}$ and hence $L_n \in \partial D_n^{+}(\lambda_0)$. From Proposition 4.38, we know $L_n \neq 0$. Hence, by Proposition 4.12, $L_n = \Omega_n^{+}(\lambda_0)$. Since any convergent subsequence of $\{\Omega_n^+(\lambda_k)\}$ *must converge to* $\Omega_n^+(\lambda_0)$ we have

$$\lim_{\lambda\to\lambda_0} \Omega_n^+(\lambda) = \Omega_n^+(\lambda_0) .$$

Ω_n^- can be treated similarly.

Proposition 4.80

Assume (r.1), (F.1)-(F.4) and (f.1) are satisfied. Let $a > 0$, $0 < \theta < \frac{\pi}{2}$ be fixed. Consider $\lambda_0 > 0$, $\xi_0 > 0$ (resp. < 0) and a sequence $(\lambda_k, \xi_k) \subset \mathcal{D}_0^+$ (resp. $\mathcal{D}_0^-$) such that

$$\lim_{k\to\infty} (\lambda_k, \xi_k) = (\lambda_0, \xi_0) . \tag{4.81}$$

Suppose furthermore that, for $k = 0,1,2,\ldots$, $v_k = U_a(\lambda_k, \xi_k, \theta, \cdot)$ is a solution of $(I)_a$ with $\lambda = \lambda_k$ in (1.1.a). Then $v_k \to v_0$ in $H^1[a,\infty) \cap L^\infty[a,\infty)$ as $k \to \infty$.

Proof

From Lemma 1.7, we know $v_k(x) \to 0$ and $v_k'(x) \to 0$ as $x \to \infty$. Thus, a similar argument to the one used to obtain (2.15) yields

$$\int_x^\infty v_k'^2(t)dt \le -v_k'(x)v_k(x) + \lambda \int_x^\infty r(t)v_k^2(t)dt . \tag{4.82}$$

From Corollary 1.56, we know

$$V_-(\lambda_k, a, \infty, x) \le v_k(x) \le V_+(\lambda_k, a, \infty, x) . \tag{4.83}$$

Put $\tilde{\lambda} = \sup_{k\ge 0} \{\lambda_k\}$. Corollary 1.17 tells us that

$$|V_\pm(\lambda_k, a, \infty, x)| \le |V_\pm(\tilde{\lambda}, a, \infty, x)| \quad \text{respectively} . \tag{4.84}$$

Therefore, letting $c(x) = \mathrm{Max}\{V_+(\tilde{\lambda}, a, \infty, x), V_-(\tilde{\lambda}, a, \infty, x)\}$, we have

$$|v_k(x)| \le c(x) \tag{4.85}$$

and clearly, by Theorem 1.2

$$\lim_{x\to\infty} c(x) = 0 . \tag{4.86}$$

From Corollary 1.72, we know

$$\|v_k'\|_{L^\infty[a,\infty)} < K_4(\lambda_k,a) < K_4(\tilde{\lambda},a) .$$

This together with (4.85) and (4.86) implies

$$\lim_{x\to\infty} |v_k(x)v_k'(x)| = 0 \quad \text{uniformly in } k .$$

Also, (4.83), (4.84) and $\int_x^\infty V_\pm^2(\tilde{\lambda},a,\infty,t)dt \to 0$ as $x \to \infty$ imply

$$\lim_{x\to\infty} \int_x^\infty v_k^2(t)dt = 0 \quad \text{uniformly in } k .$$

Therefore, from (4.82) we know

$$\lim_{x\to\infty} \int_x^\infty v_k'^2(t)dt = 0 \quad \text{uniformly in } k .$$

Thus, given $\varepsilon > 0$, if x is large

$$\int_x^\infty v_k^2(t)dt + \int_x^\infty v_k'^2(t)dt < \varepsilon \quad \text{for all } k > 0 .$$

Since $(\lambda_k,\xi_k) \to (\lambda_0,\xi_0)$, the basic theory for continuous dependence of initial value problems leads to

$$\int_a^x |v_k - v_0|^2 + |v_k' - v_0'|^2 dt < \varepsilon$$

for large k. Hence

$$\begin{aligned}\|v_k - v_0\|^2_{H^1[a,\infty)} &= \int_a^x |v_k - v_0|^2 + |v_k' - v_0'|^2 dt \\ &\quad + \int_x^\infty |v_k - v_0|^2 + |v_k' - v_0'|^2 dt \\ &< \varepsilon + 2\int_x^\infty v_k^2(t) + v_0^2(t) + v_k'^2(t) + v_0'^2(t)dt \\ &< 5\varepsilon .\end{aligned}$$

Since for any $x \in [a,\infty)$

$$
\begin{aligned}
|v_k(x) - v_0(x)|^2 &= -2 \int_x^\infty (v_k(t) - v_0(t))(v_k'(t) - v_0'(t))dt \\
&< \int_x^\infty |v_k - v_0|^2 + |v_k' - v_0'|^2 dt \\
&< \|v_k - v_0\|^2_{H^1[a,\infty)} \\
&< 5\varepsilon .
\end{aligned}
$$

We complete the proof.

Proof of Theorem 4.11

Let $u_n^{\pm}(\lambda) = U_a(\lambda,\Omega_n^{\pm}(\lambda),\cdot)$ respectively. Then the result easily follows from Lemmas 4.78 and Proposition 4.80.

§5. BIFURCATION FROM THE LOWEST POINT OF THE CONTINUOUS SPECTRUM OF THE LINEARIZED OPERATOR

In this section, our aim is to give a bifurcation result which is applicable to more general nonlinearities than those of §4, that is, $F(x,y)$ satisfies (f,2), (F.2), (F.3), (F.4) and

(F.7) There exist $\delta > 0$, $X > 0$ and functions $\psi \in C^1([0,\infty),[0,\infty))$, $w \in C^1([0,\infty),(0,\infty))$ such that $F(x,y) = \psi(w(x)|y|^\sigma)$ if $x > X$ and $|y| < \delta$. The function ψ satisfies $\psi(0) = 0$ and for $t \in (0,\infty)$, $\psi'(t) > 0$, $\psi(t) > p \cdot t^q$ for some constants $p,q > 0$. The function w satisfies

(w.3) $\dfrac{w'}{w} > \zeta > 0$ for $x \in [X,\infty)$.

It is also assumed that $r(x)$ satisfies

(r.3) $r \in C([0,\infty),(0,\infty))$, $r(x) = 1$ for $x \in [X,\infty)$.

We will show that there exist infinitely many connected components of solutions of (1.1) which are distinguished by nodal properties and these components bifurcate from the line of trivial solutions at the point $\lambda = 0$. To be more precise, we will prove

<u>Theorem 5.1</u>

Assume (r.3), (f.2), (F.2), (F.3), (F.4), (F.7) and (w.3) are satisfied. Let E be the Banach space $H^1[a,\infty) \cap L^\infty[a,\infty)$. Then, for every $n \in \mathbf{N}$, there exists an unbounded connected component C_n^+ (resp. C_n^-) $\subset [0,\infty) \times E$, emanating from $(0,0)$ such that if $(\lambda,u) \in C_n^+$ (resp. C_n^-) and $\lambda > 0$ then $u \in S_{a,n}^+(\lambda,\theta)$ (resp. $S_{a,n}^-$). Moreover, C_n^+ (resp. C_n^-) $\cap (\{\lambda\} \times E) \neq \varphi$ for every $\lambda > 0$.

Remark 5.2

Note that the assumptions of Theorem 4.11 are stronger than those of Theorem 5.1. Hence, so is the result.

Our approach is based on a method of which variants have been used in [9], [10], [13], [33], [37], [41], [42] etc. A detailed description and proof can be found in ([41], Appendix) or ([42], §3).

Proof

For fixed $a > 0$ and $0 < \theta < \frac{\pi}{2}$ we let $S_n^+(\infty) = \{(\lambda,u) \in \mathbb{R} \times E$, (λ,u) satisfies (1.1), $u > 0$ in a deleted neighborhood of $x = a$, u has exactly $n - 1$ simple zeroes in $(a,\infty)\} \cup \{(0,0)\}$, $n \geq 1$. Let $\mathcal{O}$ be any bounded open set in $\mathbb{R} \times E$ with $(0,0)$ in its interior. By ([41], Theorem A.6), it suffices to show

(i) $S_n^+(\infty) \cap \partial\mathcal{O} \neq \varphi$.

(ii) $S_n^+(\infty)$ is closed and its bounded subsets are relatively compact.

and the last assertion of the theorem. Since the latter follows from the unboundedness of C_n^+ and Corollary 1.72 we only need to prove (i) and (ii).

We prove (i) first. Let $S_n^+(b) = \{(\lambda,u) \in \mathbb{R} \times C^1[a,b]$, (λ,u) satisfies (1.4), $u > 0$ in a deleted neighborhood of $x = a$, u has exactly $n - 1$ simple zeroes in $(a,b)\} \cup \{(\mu_n(b),0)\}$ where $\mu_n(b) = \mu_n(a,b,\theta)$ was defined in (1.41.a). From a result of Rabinowitz [24], we know, for $b > a$, there exists a connected component $C_n^+(b) \subset S_n^+(b)$, containing $(\mu_n(b),0)$, such that $C_n^+(b) \cap (\{\lambda\} \times C^1[a,b]) \neq \emptyset$ for every $\lambda > \mu_n(b)$. For each $(\lambda,u) \in C_n^+(b)$, we extend u to be zero on (b,∞) and identify $C_n^+(b)$ with an unbounded connected subset of $R \times E$. Let $\{b_k\}$ be an increasing sequence such that $b_k \to +\infty$ as $k \to +\infty$. By (1.41.d), $(\mu_n(b_k),0) \in \mathcal{O}$ for all large k. Hence, there exists $(\lambda_k,u_k) \in C_n^+(b_k) \cap \partial\mathcal{O}$ for such k. Since $\mathcal{O}$ is bounded $\{\lambda_k\}$ is bounded. Using an argument analogous to the proof of Theorem 1.2, we may assume, without loss of generality that there is a $\lambda_0 \geq 0$ and $u_0 \in C^2[a,\infty) \cap H^1[a,\infty)$ such that

$$\lambda_k \to \lambda_0 , \tag{5.3}$$

$$u_k \xrightarrow{C^2} u_0 \text{ uniformly on compact subsets of } [a,\infty) . \tag{5.4}$$

Then a slightly modified version of the proof of Proposition 4.80 shows that $u_k \to u_0$ in E. Since $(\lambda_k,u_k) \in \partial\mathcal{O}$, $(\lambda_0,u_0) \in \partial\mathcal{O}$. If $\lambda_0 = 0$, Lemma 1.57, (5.3), (5.4) and (1.75.a) yield $u \equiv 0$. Since $(0,0) \in \mathcal{O}$ this would contradict $(\lambda_0,u_0) \in \partial\mathcal{O}$. Hence $\lambda_0 > 0$. To show $(\lambda_0,u_0) \in S_n^+(\infty)$ we need to prove

$$u_0 \not\equiv 0 \tag{5.5}$$

and

$$u_0 \text{ has exactly } n-1 \text{ zeroes in } (a,\infty) . \tag{5.6}$$

To show (5.5), let

$$\xi_k = \begin{cases} u_k'(a) & \text{if } \theta = 0 \\ u_k(a) & \text{if } 0 < \theta < \frac{\pi}{2} \end{cases}$$

for $k = 0,1,2,\ldots$. Since $\lambda_0 > 0$, by (5.3), $\frac{\lambda_0}{2} < \lambda_k < 2\lambda_0$ for large k. Invoking Proposition 4.38 yields $|\xi_k| > \varepsilon_{n-1}$ for such k. Hence, by (5.4), $|\xi_0| \geq \varepsilon_{n-1}$. Thus u_0 cannot be the trivial solution.

In order to prove (5.6) we recall the notation $D^+_{a,n}(\lambda,\theta)$, $Q^+_{a,n}(\lambda,\theta)$ and $U_a(\lambda,\xi,\theta,\cdot)$ introduced in §4. It suffices to show

Proposition 5.7

Suppose the hypotheses of Theorem 5.1 are satisfied. Let $a > 0$, $0 < \theta < \frac{\pi}{2}$ and $n \in \mathbf{N}$ be fixed. Let $\xi_k \in (D^+_{a,n}(\lambda_k,\theta) \cup Q^+_{a,n}(\lambda_k,\theta))$ and $(\lambda_k,\xi_k) \to (\lambda_0,\xi_0)$ as $k \to \infty$. Let $u_0 = U_a(\lambda_0,\xi_0,\theta,\cdot)$. If $\lambda_0 > 0$ and (λ_0,u_0) satisfies (1.1) then $\xi_0 \in Q^+_{a,n}(\lambda_0,\theta_0)$.

Since the proof of Proposition 5.7 needs some preliminaries we postpone it. Now we prove (ii). Let $\{(\lambda_k,u_k)\} \subset S^+_n(\infty)$ such that $(\lambda_k,u_k) \to (\lambda_0,u_0)$ in $\mathbf{R} \times E$. By the same reasoning as in (i), this implies that there exists a subsequence, still denoted by $\{(\lambda_k,u_k)\}$, such that (5.3) and (5.4) hold. If $\lambda_0 = 0$ Corollary 1.72 and (1.75.a) imply $u_0 \equiv 0$. Suppose $\lambda_0 > 0$. The same argument takes care of (5.5) together with Proposition 5.7 gives (5.6). Hence $(\lambda_0,u_0) \in S^+_n(\infty)$. So $S^+_n(\infty)$ is closed. To show the second assertion of (ii), it is sufficient to prove any bounded sequence of $S^+_n(\infty)$ contains a convergent subsequence. Let $\{(\lambda_k,u_k)\} \subset S^+_n(\infty)$ be bounded. Then the same line of reasoning as in (i) shows that there exists a subsequence, still denoted by $\{(\lambda_k,u_k)\}$, and $\lambda_0 > 0$, $u_0 \in C^2[a,\infty) \cap H^1[a,\infty)$ such that (5.3) and (5.4) hold. By Proposition 4.80, $u_k \to u_0$ in E. Thus, we complete the proof except for showing Proposition 5.7.

We begin with the preliminary work for the proof of Proposition 5.7.

Lemma 5.8

Assume the hypotheses of Tneorem 5.1 are satisfied. Given $\bar{\lambda} > 0$ there exists an $\alpha_1 = \alpha_1(\bar{\lambda})$ such that for $\lambda \in (0,\bar{\lambda}]$, if $\alpha > \alpha_1$ then any solution u satisfying

$$-u'' = \lambda r(x)u - F(x,u(x)) \ , \tag{5.9.a}$$

$$u(\alpha) = 0, \quad u(\beta) = 0 \quad (\text{resp.} \quad u \in L^2[\alpha,\infty)) \tag{5.9.b}$$

is also a solution of

$$-u'' = \lambda u - \psi(w(x)|u|^\sigma)u \ , \tag{5.10.a}$$

$$u(\alpha) = 0, \quad u(\beta) = 0 \quad (\text{resp.} \quad u \in L^2[\alpha,\infty)) \ . \tag{5.10.b}$$

Proof

By Corollary 1.72 $\|u\|_{L^\infty[\alpha,\infty)} < K_3(\lambda,\alpha)$. Since for $\lambda \in (0,\bar{\lambda}]$, $K_3(\lambda,\alpha) \to 0$ as $\alpha \to \infty$. The same arguments as those in the beginning of the proof of Theorem 3.1 give the result.

Lemma 5.11

Let c be a constant. If u is a solution satisfying the equation $-u'' = \lambda u - \psi(w(x)|u|^\sigma)u$ then $v = cu$ is a solution satisfying the equation $-v'' = \lambda v - \psi(\frac{w(x)}{|c|^\sigma}|v|^\sigma)v$.

Proof

This follows from the calculation:

$$\begin{aligned} -v'' &= -cu'' \\ &= \lambda cu - \psi(w(x)|u|^\sigma)cu \\ &= \lambda v - \psi(\frac{w(x)}{|c|^\sigma}|v|^\sigma)v \ . \end{aligned}$$

Proposition 5.12

Let $\alpha > 0$, $\lambda > 0$ and $0 < \theta < \frac{\pi}{2}$ be fixed Let $\Omega = \Omega_{\alpha,2}(\lambda,\theta)$ be the initial value (defined as in Proposition 4.12) such that

$U_\alpha(\lambda,\Omega,\theta,\cdot)$ is the unique (up to the sign) one-node solution of

$$-u'' = \lambda u - \psi(\exp(\zeta x)|u|^\sigma)u \ , \tag{5.13.a}$$

$$u(\alpha)\cos\theta - u'(\alpha)\sin\theta = 0, \quad u \in L^2[\alpha,\infty) \tag{5.13.b}$$

with the node $z = z_{\alpha,1}(\lambda,\Omega,\theta)$. Suppose the function w satisfies (w.3). Then any one-node solution which satisfies

$$-u'' = \lambda u - \psi(w(x)|u|^\sigma)u \ , \tag{5.14.a}$$

$$u(\alpha)\cos\theta - u'(\alpha)\sin\theta = 0, \quad u \in L^2[\alpha,\infty) \tag{5.14.b}$$

cannot have its node exceed z.

Remark 5.15

As mentioned in Remark 4.15 (b), we use the notation $\Omega_{\alpha,2}(\lambda,\theta)$ instead of $\Omega^+_{\alpha,2}(\lambda,\theta)$.

Proof

Let $\varepsilon > 0$ and $b = b(\varepsilon) = z + \varepsilon$. Let u_1 be the positive solution which satisfies

$$-u'' = \lambda u - \psi\left(\frac{w(b)}{\exp(\zeta b)} \exp(\zeta x)|u|^\sigma\right)u \ , \tag{5.16.a}$$

$$u(\alpha)\cos\theta - u'(\alpha)\sin\theta = 0, \quad u(b) = 0 \tag{5.16.b}$$

and u_2 be the negative solution which satisfies (5.16.a) and boundary conditions

$$u(b) = 0, \quad u \in L^2[b,\infty) \ . \tag{5.17}$$

Let u_3 and u_5 be the positive solution which satisfies (5.13.a), (5.16.b) and (5.14.a), (5.16.b) respectively. Let u_4 and u_6 be the negative solution which satisfies (5.13.a), (5.17) and (5.14.a), (5.17) respectively. Note that the existence of u_2, u_4 and u_6 follows from Theorem 1.2 and the hypotheses of ψ and w. To insure the existence of u_1, u_3 and u_5 we argue as follows. Since the restriction of $U_\alpha(\lambda,\Omega,\theta,\cdot)$ to $[\alpha,z]$ is the unique positive solution on that interval, by Proposition 1.43,

$$\lambda > \mu_1(\alpha,z,\theta) .$$

This together with $b > z$ and the fact that $\mu_1(\alpha,\cdot,\theta)$ is a decreasing function yields $\lambda > \mu_1(\alpha,b,\theta)$. Then by Proposition 1.43 again implies the existence of u_1, u_3 and u_5.

Now, from the assumption (w.3), it is easy to see that

$$w(b)\exp(\zeta x)/\exp(\zeta b) < w(x) \quad \text{if} \quad x > b$$

and

$$> w(x) \quad \text{if} \quad x < b .$$

Hence, by Corollary 1.17

$$u_5'(b) < u_1'(b) \tag{5.18.a}$$

and

$$u_6'(b) > u_2'(b) . \tag{5.18.b}$$

Next, we claim that

$$u_3'(b) < u_4'(b) . \tag{5.18.c}$$

We prove this indirectly. Note that the nonlinearity in the equation (5.13.a) satisfies the hypothesis of Corollary 4.9. Suppose $u_3'(b) = u_4'(b)$ this would imply the boundary value problem (5.13) has another one-node solution aside from $U_\alpha(\lambda,\Omega,\theta,\cdot)$ which is contrary to Corollary 4.9. If $u_3'(b) > u_4'(b)$. Since $u_4'(b) = -\Omega_{1,b}(\lambda,0)$, by Lemma 4.69

$$\xi < \Omega_{\alpha,2}(\lambda,\theta)$$

where

$$\xi = \begin{cases} u_3'(\alpha) & \text{if} \quad \theta = 0 \\ u_3(\alpha) & \text{if} \quad 0 < \theta < \frac{\pi}{2} . \end{cases}$$

On the other hand, since

$$z_{\alpha,1}(\lambda,\Omega,\theta) = z < b = z_{\alpha,1}(\lambda,\xi,\theta)$$

(4.13) yields

$$\xi > \Omega_{\alpha,2}(\lambda,\theta)$$

which contradicts an above inequality $\xi < \Omega_{\alpha,2}(\lambda,\theta)$. Therefore (5.18.c) must be true.

Let $c = (\exp(\zeta b)/w(b))^{1/\sigma}$. By Lemma 5.11,

$$u_1'(b) = cu_3'(b) \tag{5.18.d}$$

and

$$u_2'(b) = cu_4'(b) \ . \tag{5.18.e}$$

Combining (5.18.a)-(5.18.e), we obtain

$$u_6'(b) > u_5'(b) \ . \tag{5.19}$$

Thus, there is no one-node solution whose node is at b. Since this is true for any $b(\varepsilon)$ with $\varepsilon > 0$, the proof is completed.

Remark 5.20

The fact that (5.19) is true for all $b(\varepsilon)$ with $\varepsilon > 0$ will be used in the proof of Proposition 5.7.

Proof of Proposition 5.7

Suppose $U_a(\lambda_0,\xi_0,\theta,\cdot)$ has $j-1$ zeroes in (a,∞) for some $1 \leq j \leq n$. We first assume $\xi_k \in \Omega^+_{a,n}(\lambda_k,\theta)$. If $j > 1$, by the continuity of $z_{a,j-1}$

$$z_{a,j-1}(\lambda_k,\xi_k,\theta) < z_{a,j-1}(\lambda_0,\xi_0,\theta) + 1$$

for large k. Pick a sufficiently large α_1 so that

$$\alpha_1 > \max_{k\geq 1} z_{a,j-1}(\lambda_k,\xi_k,\theta) \tag{5.21}$$

and Lemma 5.8 holds for $\alpha \geq \alpha_1$. In particular, taking $\alpha = \alpha_1$ we know $V_{\tau(j)}(\lambda_k,\alpha,b,0,\cdot)$ and $V_{\tau(j+1)}(\lambda_k,b,\infty,0,\cdot)$ are solutions satisfying (5.10.a), where $b = b(\varepsilon)$ was defined in the proof of Proposition 5.12. By Proposition 5.12 and Remark 5.20 we have

$$|V'_{\tau(j)}(\lambda_k,\alpha,b,0,b)| > |V'_{\tau(j+1)}(\lambda_k,b,\infty,0,b)| \ . \qquad (5.22.a)$$

By (5.21) and (1.49)

$$|V'_{\tau(j)}(\lambda_k,z_{j-1},b,0,b)| > |V'_{\tau(j)}(\lambda_k,\alpha,b,0,b)| \qquad (5.22.b)$$

where $z_{j-1} = z_{a,j-1}(\lambda_k,\xi_k,\theta)$. Hence

$$|V'_{\tau(j)}(\lambda_k,z_{j-1},b,0,b)| > |V'_{\tau(j+1)}(\lambda_k,b,\infty,0,b)| \ . \qquad (5.22.c)$$

Since (5.22.c) is true for every $b(\varepsilon)$ with $\varepsilon > 0$ we claim that $U_a(\lambda_k,\xi_k,\theta,\cdot)$ cannot have its j-th zero exceed $z_{\alpha,1}(\lambda_k,\Omega,0)$, where $z_{\alpha,1}(\lambda_k,\Omega,0)$ and $\Omega = \Omega_{\alpha,2}(\lambda_k,0)$ were defined in Proposition 5.12. Indeed, suppose $j = n - 1$ we have

$$V'_{\tau(j)}(\lambda_k,z_{j-1},z_j,0,z_j) = V'_{\tau(j+1)}(\lambda_k,z_j,\infty,0,z_j) \qquad (5.23)$$

where $z_j = z_{a,j}(\lambda_k,\xi_k,\theta)$. Thus (5.23) would be contrary to (5.22.c) if $z_j > z_{\alpha,1}(\lambda_k,\Omega,0)$. If $j < n - 1$, we have

$$V'_{\tau(j)}(\lambda_k,z_{j-1},z_j,0,z_j) = V'_{\tau(j+1)}(\lambda_k,z_j,z_{j+1},0,z_j)$$

where $z_{j+1} = z_{a,j+1}(\lambda_k,\xi_k,\theta)$. By (1.47)

$$|V'_{\tau(j+1)}(\lambda_k,z_j,z_{j+1},0,z_j)| < |V'_{\tau(j+1)}(\lambda_k,z_j,\infty,0,z_j)| \ . \qquad (5.24)$$

Hence

$$|V'_{\tau(j)}(\lambda_k,z_{j-1},z_j,0,z_j)| < |V'_{\tau(j+1)}(\lambda_k,z_j,\infty,0,z_j)| \qquad (5.25)$$

which again would be contrary to (5.22.c) if $z_j > z_{\alpha,1}(\lambda_k,\Omega,0)$. Therefore, in either case, $U_a(\lambda_k,\xi_k,\theta,\cdot)$ cannot have its j-th zero exceed $z_{\alpha,1}(\lambda_k,\Omega,0)$. Since $z_{\alpha,1}$ is continuous in λ and ξ and by Lemma 4.78 $\Omega = \Omega_{\alpha,2}$ is continuous in λ, $z_{\alpha,1}(\cdot,\Omega_{\alpha,2}(\cdot,0),0)$ is a continuous function of λ. Since $\{\lambda_k\}$ is bounded, there exists a $\beta > a$ such that

$$\max_{k>1} z_{\alpha,1}(\lambda_k, \Omega_{\alpha,2}(\lambda_k,0),0) < \beta$$

which implies

$$\max_{k>1} z_{a,j}(\lambda_k,\xi_k,\theta) < \beta . \tag{5.26}$$

Since $\{u_k\}$ converges to u_0, uniformly in the c^1-norm, on compact subintervals of (a,∞), (5.26) implies u_0 must have at least j zeroes in (a,∞) which is contrary to our assumption that u_0 has $j - 1$ zeroes in (a,∞).

If $j = 1$, a slight modification in the above argument shows the same kind of contradiction occurs. We only sketch the significant difference: the points $z_{a,j+1}(\lambda_k,\xi_k,\theta)$ are replaced by a for all k. Thus a new version of (5.21) can be easily satisfied. Since Corollary 1.32 and (1.49) imply

$$|V'_{\tau(j)}(\lambda_k,a,b,\theta,b)| > |V'_{\tau(j)}(\lambda_k,\alpha,b,0,b)|$$

which replaces (5.22). By the same reasoning

$$|V'_{\tau(j)}(\lambda_k,a,b,\theta,b)| > |V'_{\tau(j+1)}(\lambda_k,b,\infty,0,b)| \tag{5.27}$$

which is (5.22.c) in this case.

Next, suppose $\xi_k \in D^+_{a,n}(\lambda_k,\theta)$. This case only requires a slight modification of the proof of the previous case, so we only indicate the significant differences as follows: let $b_k = z_{a,n}(\lambda_k,\xi_k,\theta)$, $k = 1,2,3,\ldots$. The functions $U_a(\lambda_k,\xi_k,\theta,\cdot)$ are understood to be defined on their maximal interval of definition. From (1.47) we know

$$|V'_{\tau(j+1)}(\lambda_k,b,\infty,0,b)| > |V'_{\tau(j+1)}(\lambda_k,b,b_k,0,b)| .$$

This together with (5.22.c) yields

$$|V'_{\tau(j)}(\lambda_k,z_{j-1},b,0,b)| > |V'_{\tau(j+1)}(\lambda_k,b,b_k,0,b)|$$

which is (5.22.c) for this case. Finally replace ∞ by b_k in (5.23)-(5.25) and (5.27).

Since any sequence $\{\xi_k\}$ contains a subsequence which lies in one of the above cases we complete the proof.

§6. ANALOGOUS RESULTS FOR RADIAL SOLUTIONS IN HIGHER DIMENSIONAL CASES

In this section, we consider the partial differential equation

$$-\Delta\hat{u} = \lambda\hat{r}(x)\hat{u}(x) - \hat{F}(x,\hat{u}(x))\hat{u}(x), \qquad x \in R^N \tag{6.1.a}$$

and seek

$$\hat{u} \in L^2(R^N) \cap C^2(R^N) . \tag{6.1.b}$$

It is assumed that $\hat{r} : R^N \to (0,\infty)$ and $\hat{F} : R^N \times R \to [0,\infty)$ are radial symmetric, that is, there exist functions $r : [0,\infty) \to (0,\infty)$ and $F : [0,\infty) \times R \to [0,\infty)$ such that

$$\hat{r}(x) = r(\rho)$$

and

$$\hat{F}(x,y) = F(\rho,y)$$

for $x \in R^N$ and $\rho = |x|$. Since our aim is to look for radial solutions of (6.1), (6.1) is equivalent to studying

$$-u'' - \frac{N-1}{\rho}u' = \lambda r(\rho)u - F(\rho,u)u, \qquad 0 < \rho < +\infty , \tag{6.2.a}$$

$$u'(0) = 0, \qquad \int_0^\infty \rho^{N-1}u^2 d\rho < +\infty \tag{6.2.b}$$

where and throughout this section prime always represent differentiation with respect to the radial variable.

Besides assuming (r.1), (F.1), (F.3), (F.4), (F.5)' (where it is understood ρ plays the role as x did in the one-dimensional case), we replace (F.2) by (F.2)'.

(F.2)' There exists positive numbers σ_i and continuous functions $\omega_i : [0,\infty) \to (0,\infty)$ which satisfying

$$\int_0^\infty \rho^{N-1}\omega_i^{-2/\sigma_i}d\rho < +\infty, \quad i = 1,2 \tag{6.3}$$

such that $F(\rho,y) > \omega_1(\rho)|y|^{\sigma_1}$ for $\rho \in [0,\infty)$, $y > 0$

and $F(\rho,y) > \omega_2(\rho)|y|^{\sigma_2}$ for $\rho \in [0,\infty)$, $y < 0$.

Remark 6.4

If we let $\hat{\omega}_i(x) = \omega_i(\rho)$ for $x \in R^N$ and $|x| = \rho$ then the growth condition (6.3) is equivalent to the condition $\int_{R^N} \hat{\omega}_i^{-2/\sigma_i}dx < \infty$ which has been imposed in [3] and [8].

Our goal is to generalize the results of §1-5 to this radial case. The arguments parallel those of the earlier sections. Therefore we will be more sketchy with details than earlier.

A new difficulty in treating problem (6.2) is that it has a singularity at the origin. Thus in the spirit of the earlier sections, we approximate (6.2) by

$$-u'' - \frac{N-1}{\rho+\varepsilon}u' = \lambda r(\rho)u - F(\rho,u)u \,, \tag{6.5.a}$$

$$u'(0) = 0, \qquad u(b) = 0 \tag{6.5.b}$$

where $\varepsilon > 0$ and $b \in (0,\infty)$. We will apply a global bifurcation result of Rabinowitz ([24], Chap. 4) as well as obtain certain estimates for solutions of (6.5). To do so, we look at the following equivalent problem:

$$(II)_{b,\varepsilon} \quad \begin{cases} -((\rho+\varepsilon)^{N-1}u')' = \lambda r(\rho)(\rho+\varepsilon)^{N-1}u - (\rho+\varepsilon)^{N-1}F(\rho,u)u \,, & (6.6.a) \\ u'(0) = 0, \qquad u(b) = 0 \,. & (6.6.b) \end{cases}$$

If $(II)_{b,\varepsilon}$ is linearized about the trivial solution $u \equiv 0$ we get

$$-((\rho+\varepsilon)^{N-1}u')' = \lambda r(\rho)(\rho+\varepsilon)^{N-1}v \,, \tag{6.7.a}$$

$$v'(0) = 0, \qquad v(b) = 0 \,. \tag{6.7.b}$$

Let $\tilde{S}^{+}_{b,\varepsilon,n}(\lambda)$ (resp. $\tilde{S}^{-}_{b,\varepsilon,n}(\lambda)$) be the set of $u \in C^2[0,b]$ such that u satisfies (6.5), $u > 0$ (resp. < 0) in a deleted neighborhood of $x = 0$ and u has exactly $n - 1$ simple zeroes in $(0,b)$, where $\lambda > 0$ and $n > 1$ is an integer. If (r.1), (F.1), (F.2)', (F.3) and (F.4) are satisfied, a direct application of the global bifurcation theorem of [24] shows that $(II)_{b,\varepsilon}$ possesses two unbounded components $C^{+}_{n}(b,\varepsilon)$ and $C^{-}_{n}(b,\varepsilon)$ of solutions in $\mathbb{R} \times C^1[0,b]$. Both contain $(\mu_n,0)$, where $\mu_n = \mu_n(b,\varepsilon)$ is the n-th eigenvalue of (6.7). If $\lambda > \mu_n$ and $(\lambda,u) \in C^{+}_{n}(b,\varepsilon)$ (resp. $C^{-}_{n}(b,\varepsilon)$), then $u \in \tilde{S}^{+}_{b,\varepsilon,n}(\lambda)$ (resp. $\tilde{S}^{-}_{b,\varepsilon,n}(\lambda)$). Furthermore, a result we mentioned in Remark 1.44 shows that the projection of $C^{+}_{n}(b,\varepsilon)$ (resp. $C^{-}_{n}(b,\varepsilon)$) on $\mathbb{R}$ is $[\mu_n,\infty)$. Also, a version of Proposition 1.43 holds here.

In order to obtain results for the limit problem, we need some estimates as follows.

<u>Lemma 6.8</u>

Assume (r.1), (F.1) and (F.2)' are satisfied. Let $\lambda > 0$ and $\varepsilon_0 > 0$. Then there exist constants $\tilde{k}_5 = \tilde{k}_5(\lambda,\varepsilon_0)$ and $\tilde{k}_6 = \tilde{k}_6(\lambda,\varepsilon_0)$ such that for all $\varepsilon \in [0,\varepsilon_0]$ and $b \in (0,\infty)$ if u is a solution of $(II)_{b,\varepsilon}$

$$\int_0^b (\rho + \varepsilon)^{N-1} u^2 d\rho < \tilde{k}_5 \qquad (6.9.a)$$

and

$$\int_0^b (\rho + \varepsilon)^{N-1} u'^2 d\rho < \tilde{k}_6 . \qquad (6.9.b)$$

Proof

We argue like the proof of (1.58.a) and (1.58.b). Let

$k_5 = k_5(\lambda,b,\varepsilon) = \sum_{i=1}^{2} (\lambda r_2)^{2/\sigma_i}(\int_0^b (\rho + \varepsilon)^{N-1}\omega_i^{-2/\sigma_i}d\rho)$ and

$k_6 = k_6(\lambda,b,\varepsilon) = (\lambda r_2)k_5(\lambda,b,\varepsilon)$. Then the same arguments as the proofs of (1.58.a) and (1.58.b), except for the presence of the extra weight $(\rho + \varepsilon)^{N-1}$, show that

$$\int_0^b (\rho + \varepsilon)^{N-1}u^2 d\rho < k_5(\lambda,b,\varepsilon) \tag{6.10.a}$$

and

$$\int_0^b (\rho + \varepsilon)^{N-1}u'^2 d\rho < K_6(\lambda,b,\varepsilon) \ . \tag{6.10.b}$$

Letting $\tilde{k}_5(\lambda,\varepsilon_0) = \sum_{i=1}^{2} (\lambda r_2)^{2/\sigma_i}(\int_0^\infty (\rho + \varepsilon_0)^{N-1}\omega_i^{-2/\sigma_i}d\rho)$ and $\tilde{k}_6(\lambda,\varepsilon_0) = (\lambda r_2)\tilde{k}_5(\lambda,\varepsilon_0)$. We have (6.9) for any $\varepsilon \in [0,\varepsilon_0]$ and $b \in (0,\infty)$.

Remark 6.11

By assumption (F.2)', it is easy to see that $\tilde{k}_i(\lambda,\varepsilon_0) < +\infty$ for any $\lambda > 0$ and $\varepsilon_0 > 0$, $i = 5,6$.

Lemma 6.12

Assume (r.1), (F.1) and (F.2)' are satisfied. Let $\lambda > 0$, $b > 1$ and $\varepsilon \in (0,\varepsilon_0)$. Let u be a solution of $(II)_{b,\varepsilon}$. Then there exists a constant $k_7 = k_7(\lambda,\varepsilon_0)$ such that

$$\|u\|_{L^\infty[0,b]} < k_7 \ . \tag{6.13}$$

Proof

Arguing like the proof of (1.57.c), we have, for $x \in [1,b]$, that

$$u^2(x) \le 2\left(\int_x^b u^2 d\rho\right)^{1/2}\left(\int_x^b u'^2 d\rho\right)^{1/2}$$

$$\le 2(x+\varepsilon)^{1-N}\left(\int_x^b (\rho+\varepsilon)^{N-1}u^2 d\rho\right)^{1/2}\left(\int_x^b (\rho+\varepsilon)^{N-1}u'^2 d\rho\right)^{1/2}$$

$$\le 2\left(\int_0^\infty (\rho+\varepsilon)^{N-1}u^2 d\rho\right)^{1/2}\left(\int_0^\infty (\rho+\varepsilon)^{N-1}u'^2 d\rho\right)^{1/2}$$

$$\le \int_0^\infty (\rho+\varepsilon)^{N-1}u^2 d\rho + \int_0^\infty (\rho+\varepsilon)^{N-1}u'^2 d\rho \ .$$

Let $k_8 = k_8(\lambda,\varepsilon_0) = (\tilde{k}_5 + \tilde{k}_6)^{1/2}$. By Lemma 6.8, we have

$$\|u\|_{L^\infty[1,b]} \le k_8 \ . \qquad (6.14)$$

Next, let $\bar{\omega}_i = \min_{\rho\in[0,1]} \omega_i$, $i = 1,2$ and $k_7 = \mathrm{Max}((\lambda r_2/\bar{\omega}_1)^{1/\sigma_1}, (\lambda r_2/\bar{\omega}_2)^{1/\sigma_2}, k_8)$. If $|u(\rho)| \le k_8$ for all $\rho \in [0,1]$ we have completed the proof. Otherwise, let $M = \max_{\rho\in[0,1]} |u(\rho)|$, $|u|$ must be equal to M at some point $t \in [0,1)$. Suppose $u(t) = M$ then u attains its maximum at t. Suppose $t = 0$. Since $u'(0) = 0$, $u''(0) \le 0$. It follows from (6.5.a) that

$$F(0,u(0)) \le \lambda r(0) \ .$$

This together with (F.2)' and (r.1) leads to

$$M \le (\lambda r_2/\omega_1(0))^{1/\sigma_1} \ .$$

If $t \in (0,1)$, then $u'(t) = 0$ and $u''(t) \le 0$. The same argument as above shows

$$M < (\lambda r_2/\omega_1(t))^{1/\sigma_1} .$$

Therefore, in either case, we have

$$M < (\lambda r_2/\bar{\omega}_1)^{1/\sigma_1} .$$

Suppose $u(t) = -M$ then u attains its minimum at t. By an analogous argument, we get

$$M < (\lambda r_2/\bar{\omega}_2)^{1/\sigma_2} .$$

Hence (6.13) easily follows.

Lemma 6.15

Assume (r.1), (F.1) and (F.2)' are satisfied. Let $\beta > 0$ be fixed. For any $b > \beta$ if u is a solution of $(II)_{b,\varepsilon}$ and $\|u\|_{L^\infty[0,\beta]} < M$ then there is a constant $k_9 = k_9(\lambda,\beta,M)$ such that for $\rho \in [0,\beta]$

$$|u'(\rho)| < k_9 \cdot \rho . \tag{6.16}$$

Proof

Integrating (6.6.a) over $[0,\rho]$ together with (6.6.b) yields

$$(\rho + \varepsilon)^{N-1}u'(\rho) = \int_0^\rho [F(t,u)u - \lambda r(t)u](t + \varepsilon)^{N-1}dt .$$

Invoking the mean value theorem for integrals, we get

$$(\rho + \varepsilon)^{N-1}u'(\rho) = A(s)(s + \varepsilon)^{N-1}\rho$$

where $A(s) = [F(s,u(s))u(s) - \lambda r(s)u(s)]$ for some $s \in [0,\rho]$.

Letting $k_9 = k_9(\lambda,\beta,M) = \underset{\substack{0<t<\beta \\ 0<y<M}}{\text{Max}} [F(t,y) + \lambda r_2]|y|$, (6.16) now easily follows.

Next, we state the uniqueness and existence result for positive (resp. negative) solutions of (6.2) as follows:

<u>Theorem 6.17</u>

Assume (r.1), (F.1), (F.4) are satisfied. Let $\lambda > 0$ be fixed. If u_1, u_2 are two solutions of (6.2) such that $u_1, u_2 > 0$ (resp. < 0), then

$$u_1 \equiv u_2 \quad \text{in} \quad [0,\infty) .$$

Let $L^2_\rho[0,\infty)$ be the weighted Hilbert space of u such that $\int_0^\infty u^2(\rho)\rho^{N-1}d\rho < \infty$. Define $H^1_\rho[0,\infty)$ by $u \in H^1_\rho[0,\infty)$ if and only if $u \in L^2_\rho[0,\infty)$ and $u' \in L^2_\rho[0,\infty)$.

<u>Theorem 6.18</u>

Assume (r.1), (F.1), (F.2)', (F.3) and (F.4) are satisfied. Given $\lambda > 0$ there exists a positive (resp. negative) function $u \in C^2[0,\infty) \cap H^1_\rho[0,\infty)$ satisfying (6.2) such that

$$\lim_{\rho\to\infty} \rho^{(N-1)/2}u(\rho) = 0 \tag{6.19.a}$$

and

$$\lim_{\rho\to\infty} \rho^{(N-1)/2}u'(\rho) = 0 . \tag{6.19.b}$$

<u>Remark 6.20</u>

(a) In fact, we will show every solution of (6.2) belongs to $C^2[0,\infty) \cap H^1_\rho[0,\infty)$ and satisfies (6.19).

(b) If $u \in C^2[0,\infty) \cap H^1_\rho[0,\infty)$ and satisfies (6.2) by letting $\hat{u}(x) = u(\rho)$ for $|x| = \rho$ and $x \in R^N$, then $\hat{u} \in C^2(R^N) \cap H^1(R^N)$ and satisfies (6.1.a).

(c) Theorem 6.17 as well as "monotonicity" properties like Corollaries 1.17, 1.32 and 1.45 actually hold for positive (resp. negative) solutions of (6.6) with $\varepsilon > 0$. Their proofs are the same as the earlier ones with only the equation (1.1.a) changed to (6.6.a) and the decay of solutions $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} u'(x) = 0$ to (6.19). Therefore we omit the proofs.

Proof of Theorem 6.18

Let $\{b_k\}$ be an increasing sequence and $\{\varepsilon_k\}$ be a decreasing sequence such that $b_k \to \infty$ and $\varepsilon_k \to 0$ as $k \to \infty$. Let $u_k = V_+(\lambda, b_k, \varepsilon_k, \cdot)$, the unique positive solution of $(II)_{b_k,\varepsilon_k}$. By Lemmas 6.12 and 6.15, we know for all $b_k > 1$, $\varepsilon_k < \varepsilon_0$, there is a constant $C_1 = C_1(\lambda, b_k, \varepsilon_0) = K_7 + K_9 \cdot b_k$, where $K_9 = K_9(\lambda, b_k, K_7)$, such that for all $\ell > k$

$$\|u_\ell\|_{C^1[0,b_k]} < C_1 ,$$

and

$$\max_{\rho\in[0,b_k]} \left|\frac{u_\ell'(\rho)}{\rho + \varepsilon}\right| < k_9 .$$

By the same line of reasoning as in the proof of Theorem 1.2, there exists a subsequence $\{u_{k_j}\}$ and a $u \in C^2[0,\infty) \cap H^1_\rho[0,\infty)$ such that

$$u_{k_j} \xrightarrow{C^2} u \text{ uniformly on compact subsets of } [0,\infty) . \tag{6.21}$$

This together with (6.16) shows $u'(0) = 0$. Also the same sort of arguments used in the proof of Proposition 1.76 shows u cannot be the trivial solution.

To show (6.19) holds, we make the transformation $v = \rho^{(N-1)/2}u$. Then (6.2.a) takes the form

$$-v'' = [\lambda r(\rho) - \frac{(N-1)(N-3)}{4\rho^2} - \mathcal{F}(\rho,v)]v \tag{6.22}$$

where $\mathcal{F}(\rho,y) = F(\rho,\rho^{(1-N)/2}y)$. Taking an $\alpha > 0$, viewing v as a solution of (6.22) on the interval $[\alpha,\infty)$, and using (6.2.b), we see $v \in L^2[\alpha,\infty)$. Invoking Lemma 1.7 yields $v(\rho) \to 0$ and $v'(\rho) \to 0$ as $\rho \to \infty$. Thus (6.6) and (6.7) follow.

Next, the existence of solutions possessing a prescribed number of nodes will also be established. Let $\tilde{S}_n^+(\lambda)$ (resp. $\tilde{S}_n^-(\lambda)$) be the set of $u \in C^2[0,\infty) \cap H_\rho^1[0,\infty)$ such that u satisfies (6.2), $u > 0$ (resp. < 0) in a deleted neighborhood of $x = 0$ and u has exactly $n - 1$ simple zeroes in $(0,\infty)$, where $\lambda > 0$ and $n > 1$ is an integer.

Theorem 6.23

Suppose (r.1), (F.1), (F.2)', (F.3), (F.4) and (F.5)' are satisfied. Let $\lambda > 0$ be given. Then $\tilde{S}_n^+(\lambda)$ and $\tilde{S}_n^-(\lambda)$ are nonempty for all $n \in \mathbb{N}$.

Proof

We first need a new version of Lemma 2.52 provided that (r.1), (F.1), (F.2)', (F.3) and (F.4) are assumed. Indeed, the proof of Lemma 2.52 depends on existence, uniqueness, continuity and "monotonicity" results for positive and negative solutions which can be insured by our hypothesis here. Next, make the transformation $v = \rho^{(N-1)/2}u$ as in (6.22). We look at the following problem

$$-v'' = \left(\lambda r(\rho) - \frac{(N-1)(N-3)}{4\rho^2}\right)v - F(\rho,v)v \ , \qquad (6.24.a)$$

$$v(\alpha) = 0, \qquad v \in L^2[\alpha,\infty) \ . \qquad (6.24.b)$$

If α is large enough, $\lambda r(\rho) - \frac{(N-1)(N-3)}{4\rho^2} > \frac{\lambda}{2} r_1$ for $\rho > \alpha$. Since the function F satisfies (F.5)' and $F(\rho,y) = F(\rho,\rho^{(1-N)/2}y)$, $F(\rho,-y) = F(\rho,y)$ if $|y| < \delta$ and $\rho > \mathrm{Max}(1,X)$. Thus the function F satisfies (F.5)'. Also $F(\rho,y)$ satisfies (F.2)' implies that $F(\rho,y)$ satisfies (F.2). The remaining assumptions of Theorem 3.1 can be checked easily. Hence, for every $n > 2$, if α is sufficiently large, by Theorem 3.1, there exists a $v_1 \in S^+_{\alpha,n}(\lambda)$ (resp. $S^-_{\alpha,n}(\lambda)$). Let $u_1 = \rho^{(1-N)/2}v_1$ and u_1 satisfies

$$-u'' - \frac{N-1}{\rho} u' = \lambda r(\rho)u - F(\rho,u)u \ , \qquad (6.25.a)$$

$$u(\alpha) = 0, \qquad u \in L^2_\rho[\alpha,\infty) \qquad (6.25.b)$$

and has the same nodes as v_1 does. The rest of the proof is like the second half of the proof of the Theorem 3.1. Let $x_0 = 0$, $x_n = \infty$ and x_i, $1 < i < n-1$, be the nodes of u_1. Invoking the "monotonicity" properties and the new version of Lemma 2.52 we take u_1 as starting point and iterate as earlier to obtain a solution which belongs to $\tilde{S}^+_n(\lambda)$ (resp. $\tilde{S}^-_n(\lambda)$).

The dimension $N = 3$ is rather special. In view of (6.22), $\frac{(N-1)(N-3)}{4\rho^2}$ vanishes if $N = 1$ or 3. This enables us to obtain more results when $N = 3$. Assuming

(F.6)' There are $\psi_1,\psi_2 \in C^1([0,\infty),[0,\infty))$, $\psi_1(0) = \psi_2(0) = 0$, $\psi_1',\psi_2' > 0$ in $(0,\infty)$ and a positive number σ such that

$$F(\rho,y) = \begin{cases} \psi_1(w(\rho)\rho^\sigma|y|^\sigma), & y > 0, \quad \rho \in [0,\infty) \\ \psi_2(w(\rho)\rho^\sigma|y|^\sigma), & y < 0, \quad \rho \in [0,\infty) \end{cases}$$

where $w \in C^1([0,\infty),[0,\infty))$.

We can invoke results from §4 to get the following uniqueness and bifurcation results for solutions of (6.2).

Theorem 6.26

Suppose (r.2), (F.6)', (ψ.1), (w.1) and (w.2) are satisfied. Let $N = 3$. Then, for very $\lambda > 0$, $\tilde{S}_n^\pm(\lambda)$ contains at most one element. If (ψ.2) is further assumed, $\tilde{S}_n^\pm$ has a unique element. Moreover, let $\tilde{E}$ be the Banach space $H_\rho^1[0,\infty) \cap L^\infty[0,\infty)$. Then, for each $n \in N$, (6.2) possesses two curves of solutions C_n^+ and C_n^- in $\mathbf{R} \times \tilde{E}$ with $C_n^\pm = \{(\lambda, u_n^\pm(\lambda)) | \lambda > 0\} \cup \{(0,0)\}$ and $u_n^\pm(\lambda) \in \tilde{S}_n^\pm$.

Remark 6.27

Theorem 6.23 is not quite applicable to the existence part of Theorem 6.26 since $F(0,y) = 0$ for $y \in \mathbf{R}$.

Proof

If $u \in \tilde{S}_n^\pm(\lambda)$, by the transformation $v = \rho u$, it is easy to see that $v \in C^2[0,\infty) \cap H^1[0,\infty)$, $v(0) = 0$, $v'(0) = u(0)$ and v satisfies (4.5). Thus, if (r.2), (F.6)', (ψ.1), (w.1) and (w.2) are satisfied, by Theorem 4.7, $\tilde{S}_n^\pm(\lambda)$ contains at most one element.

Suppose (ψ.2) is further assumed. By Corollary 4.9, $S_{0,n}^\pm(\lambda,0)$ contains a unique element $v \in C^2[0,\infty) \cap H^1[0,\infty)$. From the assumptions for the functions ψ and w, (4.5) shows $v'' \in C^1[0,\infty)$ and hence $v \in C^3[0,\infty)$. Let $u(\rho) = \rho^{-1}v(\rho)$ for $\rho \in (0,\infty)$, $u(0) = v'(0)$, $u'(0) = 0$ and $u''(0) = v'''(0)/3$. It is easy to see that $u \in C^2(0,\infty) \cap H_\rho^1[0,\infty)$ and u satisfies (6.2). We claim $u \in C^2[0,\infty)$. Indeed, $v(0) = 0$ and (4.5) imply $v''(0) = 0$. Hence

$$\lim_{\rho\to 0} u(\rho) = \lim_{\rho\to 0} \rho^{-1}v(\rho) = v'(0) .$$

So u is continuous at 0. By Taylor series expansion, $v(\rho) = v'(0)\rho + v'''(0)\rho^3/6 + o(\rho^3)$. It follows that

$$\lim_{\rho\to 0} u'(\rho) = \lim_{\rho\to 0} [\rho v'(\rho) - v(\rho)]/\rho^2 = \lim_{\rho\to 0} [v'(\rho) - v'(0)]/\rho$$

$$= v''(0) = 0 .$$

So u' is continuous at 0. Finally $v'(\rho) = v'(0) + v'''(0)\rho^2/2 + o(\rho^2)$. Thus

$$\lim_{\rho\to 0} u''(\rho) = \lim_{\rho\to 0} \frac{d}{d\rho}\left[\frac{v'(\rho) - v'(0)}{\rho} - v'''(0)\rho/6\right]$$

$$= \lim_{\rho\to 0} \left(\frac{v''(\rho)}{\rho} - \frac{v'(\rho) - v'(0)}{\rho^2} - \frac{v'''(0)}{6}\right)$$

$$= v'''(0)/3 .$$

So $u \in C^2[0,\infty)$. It is now clear that $u \in \tilde{S}_n^\pm(\lambda)$.

To prove the last assertion, we note that Theorem 4.11 and the transformation $u = \rho^{-1}v$ induce connected components of solutions of (6.2) in $\mathbb{R} \times H_\rho^1[0,\infty)$. By an argument analogous to the beginning of the proof of Lemma 6.12, we have $\|u_1 - u_2\|_{L^\infty[1,\infty)} < \|u_1 - u_2\|_{H_\rho^1[0,\infty)}$ if $(\lambda_1,u_1),(\lambda_2,u_2) \in C_n^\pm$. This together with the continuous dependence of the solution for the initial value problem

$$-u'' - \frac{N-1}{\rho}u' = \lambda r(\rho)u - F(\rho,u)u ,$$

$$u(0) = u_0, \quad u'(0) = u_0'$$

shows $C_n^\pm$ are connected in $\mathbb{R} \times \tilde{E}$.

Let $f_1(\rho,y) = F(\rho,\rho^{-1}y)y$. Under the assumptions

(f.2)' $f_1(\rho,y)$ is continuously differentiable in $[0,\infty) \times R$

and

(F.7)' There exists $\delta > 0$, $X > 0$ and functions $\psi \in C^1([0,\infty),[0,\infty))$, $w \in C^1([0,\infty),(0,\infty))$ such that $F(\rho,y) = \psi(w(\rho)\rho^{\sigma}|y|^{\sigma})$ if $\rho > X$ and $|y| < \delta$. The function ψ satisfies $\psi(0) = 0$ and for $t \in (0,\infty)$, $\psi'(t) > 0$, $\psi(t) > P \cdot t^q$ for some constants $P,q > 0$.

We have a bifurcation result which is parallel to Theorem 5.1.

<u>Theorem 6.28</u>

Suppose (r.3), (f.2)', (F.2)', (F.3), (F.4), (F.7)' and (w.3) are satisfied. Let $N = 3$ and $\tilde{E} = H^1_\rho[0,\infty) \cap L^\infty[0,\infty)$. Then, for every $n \in \mathbb{N}$, *there exists an unbounded connected component* C_n^+ (resp. C_n^-) $\subset (0,\infty) \times \tilde{E}$, emanating from $(0,0)$ such that if $(\lambda,u) \in C_n^+$ (resp. C_n^-) and $\lambda > 0$ then $u \in \tilde{S}_n^+(\lambda)$ (resp. $\tilde{S}_n^-(\lambda)$). Moreover, C_n^+ (resp. C_n^-) $\cap (\{\lambda\} \times \tilde{E}) \neq \varphi$ for every $\lambda > 0$.

The proof follows the same strategy as we used in Theorem 6.26, so we omit it.

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